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Fusion matrices, generalized Verlinde formulas and partition functions in $\mathcal{WLM}(1, p)$

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Abstract

The infinite series of logarithmic minimal models $\mathcal{LM}(1, p)$ is considered in the $\mathcal{W}$-extended picture where they are denoted by $\mathcal{WLM}(1, p)$. As in the rational models, the fusion algebra of $\mathcal{WLM}(1, p)$ is described by a simple graph fusion algebra. The corresponding fusion matrices are mutually commuting, but in general not diagonalizable. Nevertheless, they can be simultaneously brought to Jordan form by a similarity transformation. The spectral decomposition of the fusion matrices is completed by a set of refined similarity matrices converting the fusion matrices into Jordan *canonical* form consisting of Jordan blocks of rank 1, 2 or 3. The various similarity transformations and Jordan forms are determined from the modular data. This gives rise to a generalized Verlinde formula for the fusion matrices. Its relation to the partition functions in the model is discussed in a general framework. By application of a particular structure matrix and its Moore–Penrose inverse, this Verlinde formula reduces to the generalized Verlinde formula for the associated Grothendieck ring.

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1. Introduction

The fusion matrices of a standard rational conformal field theory are diagonalizable. This is made manifest by the Verlinde formula [1, 2] where the diagonalizing similarity matrix is the modular S-matrix of the characters in the theory. In a logarithmic conformal field theory, on the other hand, there are typically more linearly independent representations than linearly independent characters due to the presence of indecomposable representations of rank greater than 1. Consequently, there is no Verlinde formula in the usual sense and the fusion matrices may not all be diagonalizable. This is indeed the situation in the cases studied here, but can be circumnavigated in some logarithmic models where a Verlinde formula is recovered when restricting to a subset of the spectrum of representations and their associated characters [3]. Our results here present the first spectral decompositions of non-diagonalizable fusion matrices. Spectral decompositions of the likewise non-diagonalizable matrix realizations of the associated Grothendieck rings appear in [4], see below.

We consider the infinite series of logarithmic minimal models $\mathcal{LM}(1, p)$ [5] in the $\mathcal{W}$-extended picture [6] where they are denoted by $\mathcal{WLM}(1, p)$. The fusion rules [6–9] underlying the commutative and associative fusion algebra of $\mathcal{WLM}(1, p)$ are generated from repeated fusions of the two fundamental representations $(2, 1)_{\mathcal{W}}$ and $(1, 2)_{\mathcal{W}}$. As in rational conformal field theories [2], the fusion algebra is described by a simple graph fusion algebra. This is neatly encoded in the graphs associated with the two fundamental fusion matrices, and we exhibit these graphs explicitly.

There are $4p - 2$ indecomposable representations in the model $\mathcal{WLM}(1, p)$. According to [10], every associated fusion matrix can be written as a polynomial in the fundamental fusion matrices $\mathcal{N}_{(2,1)_{\mathcal{W}}}$ and $\mathcal{N}_{(1,2)_{\mathcal{W}}}$. We devise a similarity transformation in the form of a matrix Q which converts these two fusion matrices simultaneously into Jordan canonical form. This matrix Q is naturally described in terms of Chebyshev polynomials and derivatives thereof. It is constructed by concatenating a complete set of generalized eigenvectors of $\mathcal{N}_{(1,2)_{\mathcal{W}}}$ forming Jordan chains of length 1 or 3.

Due to the polynomial constructions just mentioned, the remaining fusion matrices are also brought to Jordan form by the similarity matrix Q, albeit typically *non-canonical* Jordan forms. The similarity matrices converting them into *canonical* Jordan forms can be obtained rather straightforwardly from Q. For every fusion matrix $\mathcal{N}$, we thus provide this modified Q-matrix $Q_{\mathcal{N}}$ as well as the corresponding Jordan canonical form of $\mathcal{N}$. Only Jordan blocks of rank 1, 2 or 3 appear in these Jordan canonical forms.

One can associate a logarithmic generalization of the Verlinde formula to the so-called Grothendieck ring of $\mathcal{WLM}(1, p)$ [4, 9]. This formula yields matrix realizations of the Grothendieck generators, of which there are $2p$, in terms of the (generalized) S-matrix. The Jordan forms of these matrices contain non-trivial Jordan blocks of rank 2, while the *fusion* matrices of $\mathcal{WLM}(1, p)$ also give rise to Jordan blocks of rank 3, as already mentioned. We also stress that there are almost twice as many representations $(4p - 2)$ than Grothendieck generators $(2p)$, the latter number being equal to the number of linearly independent characters. The coincidence of the Grothendieck ring with the fusion algebra in a standard conformal field theory therefore fails to extend to the logarithmic conformal field theory $\mathcal{WLM}(1, p)$.

Another objective of the present work is to generalize the results of [4, 9] by expressing the fusion matrices in terms of the modular data encoded in the generalized S-matrix. This yields a Verlinde-type formula for the fusion matrices themselves and not just for the generators of the associated Grothendieck ring. Other approaches to a Verlinde formula for the $\mathcal{WLM}(1, p)$ models have been proposed in [11–13]. As discussed in [4], however, they do not seem to be equivalent to the approach used in [4] which is adopted here.

Finally, we outline a general framework within which it makes sense to discuss rings of equivalence classes of fusion-algebra generators. Specializing this to the equivalence classes obtained by elevating the character identities of $\mathcal{WLM}(1, p)$ to equivalence relations between the corresponding fusion generators, we recover the Grothendieck ring of $\mathcal{WLM}(1, p)$. From the lattice description of $\mathcal{WLM}(1, p)$, the $4p - 2$ indecomposable representations mentioned above are naturally associated with boundary conditions. As already indicated, the corresponding characters are not linearly independent, but we can nevertheless talk about partition functions arising when combining two such boundary conditions. This provides a direct relationship between the generalized Verlinde formulas for the fusion algebra and the Grothendieck ring, respectively. The structure matrix, which governs the expansion of the reducible characters (of the rank-2 representations) in terms of the irreducible characters, can subsequently be used to express the Grothendieck (Verlinde) matrices in terms of the fusion (Verlinde) matrices. This explicit relation also involves the Moore–Penrose inverse of the structure matrix.

A logarithmic minimal model $\mathcal{LM}(p, p')$ is defined for every pair of relatively prime integers $1 \leqslant p < p'$ [5]. Its $\mathcal{W}$-extended picture $\mathcal{WLM}(p, p')$ is described in [6, 14, 15], including the fusion algebra of the set of indecomposable representations naturally associated with boundary conditions. In the models $\mathcal{WLM}(p, p')$ with strict inequality $1 < p$, however, there are additional irreducible representations whose fusion properties have been systematically examined only very recently [16–18]. It would be of interest to extend the work presented here and in [4, 9] on the series $\mathcal{WLM}(1, p)$ to the general series $\mathcal{WLM}(p, p')$.

2. Logarithmic minimal model $\mathcal{WLM}(1, p)$

There is a $\mathcal{W}$-extended logarithmic minimal model $\mathcal{WLM}(p, p')$ for every co-prime pair of positive integers $p < p'$ [15]. Since our interest here is in the series of these models with first label equal to 1, we write $\mathcal{WLM}(1, p)$, for simplicity, where $p > 1$.

The model $\mathcal{WLM}(1, p)$ consists of $2p$ $\mathcal{W}$-irreducible representations $(\kappa, s)_{\mathcal{W}}$ and $2p - 2$ $\mathcal{W}$-indecomposable rank-2 representations $\big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}} = \big(\mathcal{R}_{\kappa,p}^{0,b}\big)_{\mathcal{W}}$. The set of these $\mathcal{W}$-indecomposable representations is

$$\mathcal{I} = \big\{(\kappa, s)_{\mathcal{W}}, \big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}};\ \kappa \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}, b \in \mathbb{Z}_{1,p-1}\big\} \tag{2.1}$$

and has cardinality $4p - 2$. Here we have introduced

$$\mathbb{Z}_{n,m} = \mathbb{Z} \cap [n, m], \qquad n, m \in \mathbb{Z} \tag{2.2}$$

and unless otherwise specified, we let

$$\kappa, \kappa' \in \mathbb{Z}_{1,2}, \qquad s, s' \in \mathbb{Z}_{1,p}, \qquad b, b' \in \mathbb{Z}_{1,p-1}. \tag{2.3}$$

$2p$ of the $4p - 2$ $\mathcal{W}$-indecomposable representations are projective, namely the two rank-1 representations $(\kappa, p)_{\mathcal{W}}$ and all of the rank-2 representations. It follows that the two representations $(\kappa, s)_{\mathcal{W}}$ are both $\mathcal{W}$-irreducible and projective. Since we are only considering the logarithmic minimal models in the $\mathcal{W}$-extended picture, we will omit specifications such as *$\mathcal{W}$-irreducible* and simply write *irreducible* in the following.

2.1. Fusion algebra

We denote the fusion multiplication in the $\mathcal{W}$-extended picture by $\hat{\otimes}$. The fusion rules underlying the commutative and associative fusion algebra of $\mathcal{WLM}(1, p)$ read [6–9]

$$(\kappa, s)_{\mathcal{W}} \,\hat{\otimes}\, (\kappa', s')_{\mathcal{W}} = \bigoplus_{j=|s-s'|+1,\ \mathrm{by}\ 2}^{p-|p-s-s'|-1} (\kappa \cdot \kappa', j)_{\mathcal{W}} \oplus \bigoplus_{\beta=\epsilon(s+s'-p-1),\ \mathrm{by}\ 2}^{s+s'-p-1} \big(\mathcal{R}_{\kappa\cdot\kappa'}^\beta\big)_{\mathcal{W}}$$

$$(\kappa, s)_{\mathcal{W}} \,\hat{\otimes}\, \big(\mathcal{R}_{\kappa'}^b\big)_{\mathcal{W}} = \bigoplus_{\beta=|s-b|+1,\ \mathrm{by}\ 2}^{p-|p-s-b|-1} \big(\mathcal{R}_{\kappa\cdot\kappa'}^\beta\big)_{\mathcal{W}} \oplus \bigoplus_{\beta=\epsilon(s-b-1),\ \mathrm{by}\ 2}^{s-b-1} 2\big(\mathcal{R}_{\kappa\cdot\kappa'}^\beta\big)_{\mathcal{W}}$$
$$\oplus \bigoplus_{\beta=\epsilon(s+b-p-1),\ \mathrm{by}\ 2}^{s+b-p-1} 2\big(\mathcal{R}_{2\cdot\kappa\cdot\kappa'}^\beta\big)_{\mathcal{W}}$$

$$\begin{aligned}\big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}}\,\hat{\otimes}\,\big(\mathcal{R}_{\kappa'}^{b'}\big)_{\mathcal{W}} = &\bigoplus_{\beta=\epsilon(p-b-b'-1),\ \text{by } 2}^{p-|b-b'|-1} 2\big(\mathcal{R}_{\kappa\cdot\kappa'}^{\beta}\big)_{\mathcal{W}} \oplus \bigoplus_{\beta=\epsilon(p-b-b'-1),\ \text{by } 2}^{|p-b-b'|-1} 2\big(\mathcal{R}_{\kappa\cdot\kappa'}^{\beta}\big)_{\mathcal{W}}\\ &\oplus \bigoplus_{\beta=\epsilon(b+b'-1),\ \text{by } 2}^{p-|p-b-b'|-1} 2\big(\mathcal{R}_{2\cdot\kappa\cdot\kappa'}^{\beta}\big)_{\mathcal{W}} \oplus \bigoplus_{\beta=\epsilon(b+b'-1),\ \text{by } 2}^{|b-b'|-1} 2\big(\mathcal{R}_{2\cdot\kappa\cdot\kappa'}^{\beta}\big)_{\mathcal{W}},\end{aligned} \tag{2.4}$$

where we have introduced $\big(\mathcal{R}_\kappa^0\big)_{\mathcal{W}} \equiv (\kappa, p)_{\mathcal{W}}$ and

$$\epsilon(n) = \frac{1-(-1)^n}{2}, \qquad n\cdot m = 1+\epsilon(n+m) = \frac{3-(-1)^{n+m}}{2}, \qquad n, m \in \mathbb{Z}. \tag{2.5}$$

We note that this dot product is associative. The irreducible representation $(1, 1)_{\mathcal{W}}$ is the fusion-algebra identity, and the fusion algebra is seen to be generated from repeated fusions of the two fundamental representations $(2, 1)_{\mathcal{W}}$ and $(1, 2)_{\mathcal{W}}$. The works [6–9] provide considerable evidence for the validity of these fusion rules, though a rigorous proof is not known at present.

2.2. *Fusion matrices and polynomial fusion ring*

The fusion algebra, see [2] for example,

$$\phi_i \otimes \phi_j = \bigoplus_{k\in\mathcal{J}} N_{i,j}{}^k \phi_k, i, \qquad j \in \mathcal{J} \tag{2.6}$$

of a *rational* conformal field theory is finite (since the set $\mathcal{J}$ of fusion-algebra generators is finite) and can be represented by a commutative matrix algebra $\langle \mathcal{N}_i;\ i \in \mathcal{J}\rangle$ where the entries of the $|\mathcal{J}| \times |\mathcal{J}|$ matrix $\mathcal{N}_i$ are

$$(\mathcal{N}_i)_j{}^k = N_{i,j}{}^k, \qquad i, j, k \in \mathcal{J} \tag{2.7}$$

and where the fusion multiplication $\otimes$ has been replaced by ordinary matrix multiplication. In [19], Gepner found that every such algebra is isomorphic to a ring of polynomials in a finite set of variables modulo an ideal defined as the vanishing conditions of a finite set of polynomials in these variables.

With respect to some ordering of the fusion generators in (2.1), we let

$$\big\{\mathcal{N}_{(\kappa,s)_{\mathcal{W}}}, \mathcal{N}_{(\mathcal{R}_\kappa^b)_{\mathcal{W}}};\ \kappa \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}, b \in \mathbb{Z}_{1,p-1}\big\} \tag{2.8}$$

denote the set of fusion matrices realizing the fusion algebra (2.4) of $\mathcal{WLM}(1, p)$, where $\mathcal{N}_{(\kappa,s)_{\mathcal{W}}}$ and $\mathcal{N}_{(\mathcal{R}_\kappa^b)_{\mathcal{W}}}$ are the matrix realizations of the indecomposable representations $(\kappa, s)_{\mathcal{W}}$ and $\big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}}$, respectively. These are all $(4p-2)$-dimensional square matrices, and we are thus dealing with the *regular* representation of the fusion algebra. Special notation is introduced for the two fundamental fusion matrices

$$X = \mathcal{N}_{(2,1)_{\mathcal{W}}}, \qquad Y = \mathcal{N}_{(1,2)_{\mathcal{W}}}. \tag{2.9}$$

Since we are only considering $\mathcal{WLM}(1, p)$, we have thus abandoned the normalization convention of [10] and will be using the one appearing in (2.9).

From [10], we have the fusion-matrix realization

$$\begin{aligned}\mathcal{N}_{(\kappa,s)_{\mathcal{W}}} &= \text{pol}_{(\kappa,s)_{\mathcal{W}}}(X, Y) = X^{\kappa-1}U_{s-1}\left(\frac{Y}{2}\right)\\ \mathcal{N}_{(\mathcal{R}_\kappa^b)_{\mathcal{W}}} &= \text{pol}_{(\mathcal{R}_\kappa^b)_{\mathcal{W}}}(X, Y) = 2X^{\kappa-1}T_b\left(\frac{Y}{2}\right)U_{p-1}\left(\frac{Y}{2}\right)\end{aligned} \tag{2.10}$$

of the fusion algebra of $\mathcal{WLM}(1, p)$, where T_n and U_n are Chebyshev polynomials of the first and second kind, respectively. Chebyshev polynomials are ubiquitous and discussed in [20], for example, and in the appendix of [10].

It also follows from [10] that this fusion algebra is isomorphic to the polynomial ring $\mathbb{C}[X, Y]$ modulo the ideal $(X^2 - 1, P_p(Y), \tilde{P}_{1,p}(X, Y))$, that is

$$\big\langle(\kappa, s)_{\mathcal{W}}, \big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}};\ \kappa \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}, b \in \mathbb{Z}_{1,p-1}\big\rangle \simeq \mathbb{C}[X, Y]/\big(X^2 - 1, P_p(Y), \tilde{P}_{1,p}(X, Y)\big), \tag{2.11}$$

where

$$P_p(Y) = (Y^2 - 4)U_{p-1}^3\left(\frac{Y}{2}\right), \qquad \tilde{P}_{1,p}(X, Y) = \left(X - T_p\left(\frac{Y}{2}\right)\right)U_{p-1}\left(\frac{Y}{2}\right). \tag{2.12}$$

The polynomial $\tilde{P}_{1,p}(X, Y)$ differs slightly from the polynomial $P_{1,p}(X, Y)$ in [10] due to the modified normalization convention in (2.9). As demonstrated in appendix A.1, we have

$$P_p(Y) \equiv 0 \qquad (\text{mod } X^2 - 1, \tilde{P}_{1,p}(X, Y)) \tag{2.13}$$

and hence

$$\big\langle(\kappa, s)_{\mathcal{W}}, \big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}};\ \kappa \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}, b \in \mathbb{Z}_{1,p-1}\big\rangle \simeq \mathbb{C}[X, Y]/(X^2 - 1, \tilde{P}_{1,p}(X, Y)) \tag{2.14}$$

simplifying the description of the right-hand side of the isomorphism (2.11).

It is noted that X and Y in (2.11) and (2.14) are formal entities and hence need not be identified with the fusion matrices X and Y in (2.9) and (2.10). It is nevertheless convenient to use the same notation in the two situations. Using the explicit fusion matrices and their Jordan decompositions to be discussed below, the quotient polynomial ring conditions in (2.11) are verified partly by (3.7) and otherwise in appendix A.2.

3. Explicit fusion matrices

The set of fusion generators (2.1) is distinguished in the sense that the associated fusion rules (2.4) involve only non-negative integer multiplicities. It turns out that the ordering

$$\begin{aligned}&(1, 1)_{\mathcal{W}}, (2, 1)_{\mathcal{W}}; \ldots; (1, s)_{\mathcal{W}}, (2, s)_{\mathcal{W}}; \ldots; (1, p)_{\mathcal{W}}, (2, p)_{\mathcal{W}};\\ &\qquad\big(\mathcal{R}_1^1\big)_{\mathcal{W}}, \big(\mathcal{R}_2^1\big)_{\mathcal{W}}; \ldots; \big(\mathcal{R}_1^b\big)_{\mathcal{W}}, \big(\mathcal{R}_2^b\big)_{\mathcal{W}}; \ldots; \big(\mathcal{R}_1^{p-1}\big)_{\mathcal{W}}, \big(\mathcal{R}_2^{p-1}\big)_{\mathcal{W}}\end{aligned} \tag{3.1}$$

provides a convenient basis in which to study the fusion matrices, and is the one used in the following. It is recalled that we are working with the regular representation of the fusion algebra.

It should be clear that

$$\mathcal{N}_{(1,1)_{\mathcal{W}}} = I. \tag{3.2}$$

The fusion matrices X and Y are given in (3.5) below. The fusion-matrix realizations of the remaining generators in (2.1) can all be obtained by polynomial constructions (2.10) from the realizations X and Y of the fundamental representations $(2, 1)_{\mathcal{W}}$ and $(1, 2)_{\mathcal{W}}$.

3.1. Fundamental fusion matrices

To facilitate the description of the fundamental fusion matrices, we introduce

$$0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

For simplicity, we use $I = I_m$ and $C = C_{2n}$ to denote the m-dimensional identity matrix and the $2n$-dimensional square matrix

$$C = \text{diag}(\underbrace{C_2, \ldots, C_2}_{n}), \tag{3.4}$$

respectively, when their dimensions are understood from the context. We note that C is an involutory matrix, $C^2 = I$.

In the basis (3.1), the fundamental fusion matrices read

$$
X = C, \qquad Y = \left(\begin{array}{cccccc|c|cccccc}
0_2 & I_2 & 0_2 & \cdots & & & & & & & & & \\
I_2 & 0_2 & I_2 & \ddots & & & & & & & & & \\
0_2 & I_2 & 0_2 & \ddots & & & & & & & & & \\
\vdots & \ddots & \ddots & \ddots & & & & & & & & & \\
& & & & 0_2 & I_2 & & & & & & & \\
& & & & I_2 & 0_2 & I_2 & & & & & & \\
\hline
& & & & & 0_2 & 0_2 & I_2 & & & & & \\
\hline
& & & & & & 2I_2 & 0_2 & I_2 & & & & \\
& & & & & & & I_2 & 0_2 & I_2 & & & \\
& & & & & & & & I_2 & 0_2 & \ddots & & \\
& & & & & & & & & \ddots & \ddots & & \\
& & & & & & & & & & & 0_2 & I_2 \\
& & & & & & 2C_2 & & & & & I_2 & 0_2
\end{array}\right). \tag{3.5}
$$

The $(4p-2)$-dimensional matrix Y is written here as a $(2p-1)$-dimensional matrix, with 2×2 matrices as entries, whose pth row and column are emphasized to indicate their special status. For small values of p, expression (3.5) for Y is meant to reduce to

$$
Y|_{p=2} = \begin{pmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ 0_2 & 2I_2 + 2C_2 & 0_2 \end{pmatrix}, \qquad Y|_{p=3} = \begin{pmatrix} 0_2 & I_2 & 0_2 & 0_2 & 0_2 \\ I_2 & 0_2 & I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & 2I_2 & 0_2 & I_2 \\ 0_2 & 0_2 & 2C_2 & I_2 & 0_2 \end{pmatrix}. \tag{3.6}
$$

As required, it follows from (3.5) that X and Y commute.

The minimal and characteristic polynomials of X are readily seen to be

$$
X^2 - I = (X - I)(X + I), \qquad \det(\lambda I - X) = (\lambda - 1)^{2p-1}(\lambda + 1)^{2p-1}. \tag{3.7}
$$

The description of the similar polynomials for Y uses that the Chebyshev polynomial $U_{p-1}(x)$ factorizes as

$$
U_{p-1}(x) = 2^{p-1} \prod_{j=1}^{p-1} (x - \alpha_j), \qquad \alpha_j = \cos\theta_j, \qquad \theta_j = \frac{j\pi}{p}, \qquad j \in \mathbb{Z}_{0,p}, \tag{3.8}
$$

where we have included definitions of $\alpha_0 = 1$ and $\alpha_p = -1$. In accordance with (2.12), and as we shall verify explicitly in appendix A.2, the minimal and characteristic polynomials of Y are

$$
\begin{aligned}
P_p(Y) &= (Y^2 - 4I)U_{p-1}^3\left(\frac{Y}{2}\right) = (Y - 2I)(Y + 2I)\prod_{j=1}^{p-1}(Y - 2\alpha_j I)^3 \\
\det(\lambda I - Y) &= U_{p-1}\left(\frac{\lambda}{2}\right) P_p(\lambda) = (\lambda^2 - 4)U_{p-1}^4\left(\frac{\lambda}{2}\right) = (\lambda - 2)(\lambda + 2)\prod_{j=1}^{p-1}(\lambda - 2\alpha_j)^4.
\end{aligned} \tag{3.9}
$$

This implies that the Jordan canonical form of Y consists of $p-1$ rank-3 blocks associated with the eigenvalues $\beta_b = 2\alpha_b$, $b \in \mathbb{Z}_{1,p-1}$ and $p+1$ rank-1 blocks associated with the eigenvalues

$\beta_j = 2\alpha_j$, $j \in \mathbb{Z}_{0,p}$. The number of linearly independent eigenvectors of Y is thus $2p$. Since the null space of Y is empty for p odd but two-dimensional for p even, the rank of Y is

$$\mathrm{rank}(Y) = 4p - 2 - 2\epsilon(p-1) = 4(p-1) + 2\epsilon(p). \tag{3.10}$$

4. Fusion graphs

The fusion matrices (2.8) are mutually commuting, but in general not diagonalizable. Nevertheless, we will show that they can be simultaneously brought to Jordan form by a similarity transformation, and that the associated similarity matrix is determined from the modular data. Prior to demonstrating these important results, we here discuss the two graphs whose underlying adjacency matrices are given by the fundamental fusion matrices $X = \mathcal{N}_{(2,1)_{\mathcal{W}}}$ and $Y = \mathcal{N}_{(1,2)_{\mathcal{W}}}$. In this context,

$$\mathcal{N}_\mu \mathcal{N}_\nu = \sum_{\lambda \in \mathcal{I}} \mathcal{N}_{\mu,\nu}{}^{\lambda} \mathcal{N}_\lambda \tag{4.1}$$

is referred to as the *graph fusion algebra*. Here $\mathcal{I}$ is the set of indecomposable representations given in (2.8). To simplify the notation, we introduce

$$N_{\kappa,s} = \mathcal{N}_{(\kappa,s)_{\mathcal{W}}}, \qquad N_\kappa^b = \mathcal{N}_{(\mathcal{R}_\kappa^b)_{\mathcal{W}}}. \tag{4.2}$$

Fusion graphs succinctly encode the fusion rules and have been instrumental in the classification of rational conformal field theories on the cylinder [21, 22] and on the torus [23–26]. In the rational A-type theories, the Verlinde algebra yields a diagonal modular invariant, while D- and E-type theories are related to non-diagonal modular invariants. The Ocneanu algebras arise when considering fusion on the torus, with left and right chiral halves of the theory, and involve Ocneanu graphs. We refer to [27–30] for earlier results on the inter-relation between fusion algebras, graphs and modular invariants.

The fundamental fusion graph associated with Y follows from (3.5). For $p = 4$, in particular, it is given by

$$\begin{tikzcd}
&&&&& N_1^2 \arrow[dr, leftrightarrow] &&&&& \\
&&&& N_1^1 \arrow[ur, leftrightarrow] && N_1^3 \arrow[dr] &&&& \\
N_{1,1} \arrow[r, leftrightarrow] & N_{1,2} \arrow[r, leftrightarrow] & N_{1,3} \arrow[r] & N_{1,4} \arrow[ur, leftrightarrow] &&&& N_{2,4} & N_{2,3} \arrow[l] \arrow[r, leftrightarrow] & N_{2,2} \arrow[r, leftrightarrow] & N_{2,1} \\
&&&& N_2^3 \arrow[ul] \arrow[dr, leftrightarrow] && N_2^1 \arrow[ur, leftrightarrow] &&&& \\
&&&&& N_2^2 \arrow[ur, leftrightarrow] &&&&&
\end{tikzcd} \tag{4.3}$$

This is readily extended to general p where we have

$$\begin{tikzcd}
&&&&& \cdots \arrow[dr, leftrightarrow] &&&&& \\
&&&& N_1^1 \arrow[ur, leftrightarrow] && N_1^{p-1} \arrow[dr] &&&& \\
N_{1,1} \arrow[r, leftrightarrow] & \ldots \arrow[r, leftrightarrow] & N_{1,p-1} \arrow[r] & N_{1,p} \arrow[ur, leftrightarrow] &&&& N_{2,p} & N_{2,p-1} \arrow[l] \arrow[r, leftrightarrow] & \ldots \arrow[r, leftrightarrow] & N_{2,1} \\
&&&& N_2^{p-1} \arrow[ul] \arrow[dr, leftrightarrow] && N_2^1 \arrow[ur, leftrightarrow] &&&& \\
&&&&& \cdots \arrow[ur, leftrightarrow] &&&&&
\end{tikzcd} \tag{4.4}$$

which, for $p = 2$, reduces to

$$\begin{tikzcd} & & N_1^1 \arrow[dl, <->] \arrow[dr] & & \\ N_{1,1} \arrow[r] & N_{1,2} & & N_{2,2} & N_{2,1} \arrow[l] \\ & & N_2^1 \arrow[ul] \arrow[ur, <->] & & \end{tikzcd} \tag{4.5}$$

The fundamental fusion graph associated with X has $2p - 1$ disconnected components

$$N_{1,s} \longleftrightarrow N_{2,s} \qquad\qquad N_1^b \longleftrightarrow N_2^b \tag{4.6}$$

where it is recalled that $s \in \mathbb{Z}_{1,p}$ and $b \in \mathbb{Z}_{1,p-1}$. The fundamental fusion graph associated with Y is covariant under the action of X in the sense that X acts by rotating the graph, as it is depicted in (4.4), by 180°. Ignoring the labeling of the N's appearing in (4.4), the graph itself is invariant under rotation by 180°.

It is recalled that there are $2p$ irreducible representations, $(\kappa, s)_{\mathcal{W}}$, and $2p$ projective representations, $(\kappa, p)_{\mathcal{W}}$ and $\big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}}$, where the two representations $(\kappa, p)_{\mathcal{W}}$ are both irreducible and projective. The two horizontal legs of the fusion graph (4.4) are composed of the irreducible representations, while the loop consists of the projective representations with $N_{\kappa,p} \sim (\kappa, p)_{\mathcal{W}}$ appearing in both a horizontal leg and the loop. Combined with the two one-way arrows linking $N_{\kappa,p-1} \to N_{\kappa,p}$, this reflects that the set of projective representations forms an ideal of the fusion algebra (2.4).

5. Spectral decompositions

In preparation for the spectral decomposition of the various fusion matrices, we recall that the canonical rank-3 Jordan block associated with the eigenvalue λ is given by

$$\mathcal{J}_{\lambda,3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{5.1}$$

It is sometimes convenient to relax the condition of unity in the super-diagonal of a Jordan block. We thus refer to any matrix of the form

$$\begin{pmatrix} \lambda & \nu_1 & \nu_3 \\ 0 & \lambda & \nu_2 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad\qquad \nu_1 \neq 0, \ \ \nu_2 \neq 0, \tag{5.2}$$

as a Jordan block of rank-3 associated with the eigenvalue λ. The Jordan *canonical* block $\mathcal{J}_{\lambda,3}$ is recovered by setting $\nu_1, \nu_2 = 1$ and $\nu_3 = 0$. A block-diagonal matrix consisting of (canonical) Jordan blocks only is said to be in Jordan (canonical) form. Also, for a function g expandable as a power series, we note that

$$g\left(\begin{pmatrix} \lambda & \nu_1 & \nu_3 \\ 0 & \lambda & \nu_2 \\ 0 & 0 & \lambda \end{pmatrix}\right) = \begin{pmatrix} g(\lambda) & \nu_1 g'(\lambda) & \nu_3 g'(\lambda) + \frac{1}{2}\nu_1\nu_2 g''(\lambda) \\ 0 & g(\lambda) & \nu_2 g'(\lambda) \\ 0 & 0 & g(\lambda) \end{pmatrix}. \tag{5.3}$$

Our first objective in this section is to devise a similarity transformation in the form of a matrix Q which Jordan-decomposes X and Y simultaneously:

$$Q^{-1} X Q = J_X, \qquad\qquad Q^{-1} Y Q = J_Y, \tag{5.4}$$

where J_X and J_Y are Jordan canonical forms. For every $\mathcal{N}$ in (2.8), it then follows that

$$Q^{-1}\mathcal{N}Q = Q^{-1}\mathrm{pol}_{\mathcal{N}}(X, Y)Q = \mathrm{pol}_{\mathcal{N}}(Q^{-1}XQ, Q^{-1}YQ) = \mathrm{pol}_{\mathcal{N}}(J_X, J_Y), \tag{5.5}$$

implying that Q also brings $\mathcal{N}$ to Jordan form, albeit not necessarily Jordan *canonical* form. Our second objective is therefore to find invertible matrices $\hat{Q}_{\mathcal{N}}$ such that

$$Q_{\mathcal{N}} = Q\hat{Q}_{\mathcal{N}} \tag{5.6}$$

converts $\mathcal{N}$ into

$$Q_{\mathcal{N}}^{-1}\mathcal{N}Q_{\mathcal{N}} = \hat{Q}_{\mathcal{N}}^{-1}\mathrm{pol}_{\mathcal{N}}(J_X, J_Y)\hat{Q}_{\mathcal{N}} = J_{\mathcal{N}}, \tag{5.7}$$

where $J_{\mathcal{N}}$ is a Jordan canonical form.

5.1. *Fundamental fusion matrices*

For $k \in \mathbb{Z}_{1,2p-1}$, we define the function $f_k(x)$ by

$$f_s(x) = U_{s-1}\left(\frac{x}{2}\right), \qquad f_{p+b}(x) = 2T_b\left(\frac{x}{2}\right)U_{p-1}\left(\frac{x}{2}\right) = U_{p+b-1}\left(\frac{x}{2}\right) + U_{p-b-1}\left(\frac{x}{2}\right), \tag{5.8}$$

where it is recalled that $s \in \mathbb{Z}_{1,p}$ and $b \in \mathbb{Z}_{1,p-1}$. These functions describe the Y-parts of (2.10):

$$\mathcal{N}_{(\kappa,s)_{\mathcal{W}}} = X^{\kappa-1}f_s(Y), \qquad \mathcal{N}_{(\mathcal{R}_\kappa^b)_{\mathcal{W}}} = X^{\kappa-1}f_{p+b}(Y) \tag{5.9}$$

and are used below in the construction of the similarity matrix Q. Certain properties of $f_k(x)$ are described in appendix B. In the following, we will initially assume that $p > 2$ and subsequently consider the case $p = 2$.

5.1.1. *$p > 2$.*

Let us introduce the $p + 1$ $(4p - 2)$-dimensional vectors

$$v_0 = \begin{pmatrix} f_1(\beta_0)\mathbf{v}_0 \\ \vdots \\ f_{2p-1}(\beta_0)\mathbf{v}_0 \end{pmatrix}, \qquad v_b = \begin{pmatrix} f_1(\beta_b)\mathbf{v}_{b-1} \\ \vdots \\ f_{2p-1}(\beta_b)\mathbf{v}_{b-1} \end{pmatrix}, \qquad v_p = \begin{pmatrix} f_1(\beta_p)\mathbf{v}_p \\ \vdots \\ f_{2p-1}(\beta_p)\mathbf{v}_p \end{pmatrix}, \tag{5.10}$$

where we draw attention to the different conventions for the indices of the auxiliary two-dimensional vectors

$$\mathbf{v}_n = \begin{pmatrix} 1 \\ (-1)^n \end{pmatrix}, \qquad n \in \mathbb{Z}. \tag{5.11}$$

For every $b \in \mathbb{Z}_{1,p-1}$, we also introduce the triplet of $(4p - 2)$-dimensional vectors:

$$w_b^{(1)} = \begin{pmatrix} f_1(\beta_b)\mathbf{v}_b \\ \vdots \\ f_{2p-1}(\beta_b)\mathbf{v}_b \end{pmatrix}, \qquad w_b^{(2)} = \begin{pmatrix} f_1'(\beta_b)\mathbf{v}_b \\ \vdots \\ f_{2p-1}'(\beta_b)\mathbf{v}_b \end{pmatrix}, \qquad w_b^{(3)} = \begin{pmatrix} \frac{1}{2}f_1''(\beta_b)\mathbf{v}_b \\ \vdots \\ \frac{1}{2}f_{2p-1}''(\beta_b)\mathbf{v}_b \end{pmatrix}. \tag{5.12}$$

Using (B.1)–(B.3), in particular, it is straightforward to verify that the $p + 1$ vectors in (5.10) are eigenvectors of Y corresponding to the eigenvalues β_0, β_b and β_p, and that, for every $b \in \mathbb{Z}_{1,p-1}$, the three vectors in (5.12) form a Jordan chain

$$Yw_b^{(3)} = \beta_b w_b^{(3)} + w_b^{(2)}, \qquad Yw_b^{(2)} = \beta_b w_b^{(2)} + w_b^{(1)}, \qquad Yw_b^{(1)} = \beta_b w_b^{(1)} \tag{5.13}$$

corresponding to the eigenvalue β_b. This chain of relations imply that

$$(Y-\beta_b I)w_b^{(3)}=w_b^{(2)}, \qquad (Y-\beta_b I)w_b^{(2)}=w_b^{(1)}, \qquad (Y-\beta_b I)^\ell w_b^{(\ell)}=0, \quad \ell\in\mathbb{Z}_{1,3}, \tag{5.14}$$

where the vanishing conditions indicate that the vectors are generalized eigenvectors.

The two eigenvectors v_b and $w_b^{(1)}$ correspond to the same eigenvalue β_b but are obviously linearly independent. Since $\beta_i\neq\beta_j$ for $i\neq j$, the $(4p-2)$-dimensional matrix Q is constructed by concatenating the generalized (of which $2p$ are proper) eigenvectors (5.10) and (5.12):

$$Q=\left(v_0 \mid v_1 \quad w_1^{(1)} \quad w_1^{(2)} \quad w_1^{(3)} \mid \ldots \mid v_{p-1} \quad w_{p-1}^{(1)} \quad w_{p-1}^{(2)} \quad w_{p-1}^{(3)} \mid v_p\right). \tag{5.15}$$

By the similarity transformation (5.4), this matrix Q converts Y into its Jordan canonical form

$$J_Y=\mathrm{diag}(\beta_0;\beta_1,\mathcal{J}_1;\ldots;\beta_b,\mathcal{J}_b;\ldots;\beta_{p-1},\mathcal{J}_{p-1};\beta_p), \qquad \mathcal{J}_b=\mathcal{J}_{\beta_b,3,} \tag{5.16}$$

where $\mathcal{J}_{\beta_b,3}$ is the canonical rank-3 Jordan block (5.1) associated with the eigenvalue β_b. Thus, the eigenvalues $\beta_0=2$ and $\beta_p=-2$ both have geometric and algebraic multiplicity 1, whereas the $p-1$ eigenvalues $\beta_b=2\cos\theta_b$ all have geometric multiplicity 2 and algebraic multiplicity 4. This is in accordance with the minimal and characteristic polynomials of Y in (3.9).

It is readily verified that the Jordan canonical form (5.4) of X with respect to Q in (5.15) is the diagonal matrix

$$J_X=\mathrm{diag}(1;1,-I_3;\ldots;(-1)^{b-1},(-1)^bI_3;\ldots;(-1)^{p-2},(-1)^{p-1}I_3;(-1)^p). \tag{5.17}$$

The eigenvalues 1 and -1 both appear with geometric and algebraic multiplicity $2p-1$ in accordance with the minimal and characteristic polynomials in (3.7).

In conclusion, the matrix Q (5.15) converts the fundamental fusion matrices X and Y simultaneously into their Jordan canonical forms J_X and J_Y. We note that J_X and J_Y commute since X and Y commute. Their commutativity also follows directly from their compatible block structures.

As an aside, based on explicit evaluations of the determinant of Q for small values of p, we conjecture that, for general p, it is given by

$$\det Q=32(-1)^p(2p)^{5p-9}. \tag{5.18}$$

5.1.2. $p=2$. For $p=2$, Y is given in (3.6) and the expressions (5.10) and (5.12) yield the six generalized eigenvectors

$$v_0=\begin{pmatrix}1\\1\\2\\2\\4\\4\end{pmatrix}, \qquad v_1=\begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \qquad w_1^{(1)}=\begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix},$$
$$w_1^{(2)}=\begin{pmatrix}0\\0\\1\\-1\\0\\0\end{pmatrix}, \qquad w_1^{(3)}=\begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix}, \qquad v_2=\begin{pmatrix}1\\1\\-2\\-2\\4\\4\end{pmatrix} \tag{5.19}$$

corresponding to the eigenvalues $\beta_0 = -2$, $\beta_1 = 0$ and $\beta_2 = 2$. The associated similarity matrix (5.15) reads

$$Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & -2 \\ 2 & 0 & 0 & -1 & 0 & -2 \\ 4 & 0 & 0 & 0 & 1 & 4 \\ 4 & 0 & 0 & 0 & -1 & 4 \end{pmatrix} \tag{5.20}$$

and has determinant $\det(Q) = 128$ (in accordance with (5.18)), and Jordan decomposes X and Y simultaneously

$$Q^{-1}XQ = \mathrm{diag}(1, 1, -1, -1, -1, 1), \qquad Q^{-1}YQ = \mathrm{diag}(2, 0, \mathcal{J}_1, -2), \tag{5.21}$$

where the diagonal elements of the canonical rank-3 Jordan block $\mathcal{J}_1$ are $\beta_1 = 0$.

5.2. *General fusion matrices*

The similarity matrix Q brings all fusion matrices $\mathcal{N}$ simultaneously to Jordan form (5.5). Except for the two fundamental fusion matrices X and Y, these Jordan forms are typically *non-canonical*. The objective here is to determine the refinement $Q_{\mathcal{N}}$ (5.6) of Q converting the fusion matrix $\mathcal{N}$ into the Jordan *canonical* form (5.7). Thus, continuing the Jordan decomposition of $\mathcal{N}$ in (5.5) and (5.7), we have

$$\begin{aligned} Q_{\mathcal{N}}^{-1}\mathcal{N}Q_{\mathcal{N}} &= \hat{Q}_{\mathcal{N}}^{-1}J_X^{\kappa-1}f(J_Y)\hat{Q}_{\mathcal{N}} \\ &= \hat{Q}_{\mathcal{N}}^{-1}\mathrm{diag}\big(f(\beta_0); \ldots; (-1)^{(\kappa-1)(b-1)}f(\beta_b), (-1)^{(\kappa-1)b}f(\mathcal{J}_b); \ldots; \\ &\quad (-1)^{(\kappa-1)p}f(\beta_b)\big)\hat{Q}_{\mathcal{N}}, \end{aligned} \tag{5.22}$$

where b runs from 1 to $p-1$, while $f = f_k$, $k \in \mathbb{Z}_{1,2p-1}$, is the function partaking in the description of the given fusion matrix $\mathcal{N}$ (5.9). With (5.3) in mind, we here list the Jordan decompositions of all three-dimensional upper-triangular matrices whose entries of a given (super-)diagonal are identical:

$$\begin{aligned} &\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{a} & & \\ & \mathrm{a} & \\ & & \mathrm{a} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{a} & & \\ & \mathrm{a} & \\ & & \mathrm{a} \end{pmatrix} \\ &\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\mathrm{c}} \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{a} & 0 & \mathrm{c} \\ & \mathrm{a} & 0 \\ & & \mathrm{a} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} \mathrm{a} & 0 & 0 \\ & \mathrm{a} & 1 \\ & & \mathrm{a} \end{pmatrix}, \qquad \mathrm{c} \neq 0 \\ &\begin{pmatrix} 1 & \frac{\mathrm{c}}{\mathrm{b}^2} & 0 \\ & \frac{1}{\mathrm{b}} & 0 \\ & & \frac{1}{\mathrm{b}^2} \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ & \mathrm{a} & \mathrm{b} \\ & & \mathrm{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{\mathrm{c}}{\mathrm{b}^2} & 0 \\ & \frac{1}{\mathrm{b}} & 0 \\ & & \frac{1}{\mathrm{b}^2} \end{pmatrix} = \begin{pmatrix} \mathrm{a} & 1 & 0 \\ & \mathrm{a} & 1 \\ & & \mathrm{a} \end{pmatrix}, \qquad \mathrm{b} \neq 0. \end{aligned} \tag{5.23}$$

To complete the spectral decomposition of the fusion matrix $\mathcal{N}$, by finding the associated Jordan canonical form $J_{\mathcal{N}}$ and similarity matrix $\hat{Q}_{\mathcal{N}}$, it is therefore necessary to determine whether $f_k'(\beta_b)$ or $f_k''(\beta_b)$ is zero for $k \in \mathbb{Z}_{1,2p-1}$ and $b \in \mathbb{Z}_{1,p-1}$. These possibilities are classified in appendix B.3.

Now, the results (B.13) immediately confirm that

$$J_{(1,1)_{\mathcal{W}}} = I, \qquad J_{(2,1)_{\mathcal{W}}} = J_X, \qquad J_{(1,2)_{\mathcal{W}}} = J_Y \tag{5.24}$$

and

$$Q_{(1,1)_\mathcal{W}} = Q_{(2,1)_\mathcal{W}} = Q_{(1,2)_\mathcal{W}} = Q, \qquad \hat{Q}_{(1,1)_\mathcal{W}} = \hat{Q}_{(2,1)_\mathcal{W}} = \hat{Q}_{(1,2)_\mathcal{W}} = I, \tag{5.25}$$

where we have introduced the simplified notation $J_{(\kappa,s)_\mathcal{W}} = J_{\mathcal{N}_{(\kappa,s)_\mathcal{W}}}$ and similarly for $Q_\mathcal{N}$ and $\hat{Q}_\mathcal{N}$. Below, we will also use $J_{(\mathcal{R}_\kappa^b)_\mathcal{W}} = J_{\mathcal{N}_{(\mathcal{R}_\kappa^b)_\mathcal{W}}}$ and similarly for $Q_\mathcal{N}$ and $\hat{Q}_\mathcal{N}$. The Jordan canonical forms $J_\mathcal{N}$ and similarity matrices $\hat{Q}_\mathcal{N}$ of the remaining (cf (5.24) and (5.25)) fusion matrices $\mathcal{N}_{(\kappa,s)_\mathcal{W}}$ and $\mathcal{N}_{(\mathcal{R}_\kappa^{b'})_\mathcal{W}}$ in (2.8) depend on the relations between p, the labels κ, s and b', and the labeling b of the eigenvalues β_b. In the following, we list these results for $J_\mathcal{N}$ and $\hat{Q}_\mathcal{N}$, recalling that the similarity matrix Jordan-decomposing $\mathcal{N}$ is given by $Q_\mathcal{N} = Q\hat{Q}_\mathcal{N}$ (5.6). To this end, we introduce

$$\mathcal{Q}_{g(x)} = \begin{pmatrix} 1 & \frac{g'(x)}{2g^2(x)} & 0 \\ 0 & \frac{1}{g(x)} & 0 \\ 0 & 0 & \frac{1}{g^2(x)} \end{pmatrix}, \qquad \mathcal{J}_{\lambda,3}^{(1,2)} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \tag{5.26}$$

where g is a polynomial evaluated at a point where $g(x) \neq 0$.

Now, for p odd, we have $f_k'(\beta_b) \neq 0$ for $k \in \mathbb{Z}_{2,2p-1}$, implying that, for $s \in \mathbb{Z}_{2,p}$,

$$\begin{aligned} J_{(\kappa,s)_\mathcal{W}} &= \mathrm{diag}(s; \ldots; (-1)^{(\kappa-1)(b-1)} f_s(\beta_b), \mathcal{J}_{(-1)^{(\kappa-1)b} f_s(\beta_b),3}; \ldots; s(-1)^{\kappa+s}) \\ J_{(\kappa,p)_\mathcal{W}} &= \mathrm{diag}(p; \ldots; 0, \mathcal{J}_{0,3}; \ldots; p(-1)^{\kappa+1}) \\ J_{(\mathcal{R}_\kappa^{b'})_\mathcal{W}} &= \mathrm{diag}(2p; \ldots; 0, \mathcal{J}_{0,3}; \ldots; 2p(-1)^{\kappa+b'-1}) \\ \hat{Q}_\mathcal{N} &= \mathrm{diag}(1; \ldots; 1, \mathcal{Q}_{(-1)^{(\kappa-1)b} f'(\beta_b)}; \ldots; 1), \end{aligned} \tag{5.27}$$

where $f = f_s$ for $\mathcal{N} = \mathcal{N}_{(\kappa,s)_\mathcal{W}}$ while $f = f_{p+b'}$ for $\mathcal{N} = \mathcal{N}_{(\mathcal{R}_\kappa^{b'})_\mathcal{W}}$. For p even, we let $b_1 \in \mathbb{Z}_{1,\frac{p}{2}-1}$ and $b_2 \in \mathbb{Z}_{\frac{p}{2}+1,p-1}$ and find

$$\begin{aligned} J_{(\kappa,s)_\mathcal{W}} &= \mathrm{diag}\big(s; \ldots; (-1)^{(\kappa-1)(b_1-1)} f_s(\beta_{b_1}), \mathcal{J}_{(-1)^{(\kappa-1)b_1} f_s(\beta_{b_1}),3}; \ldots; 0, \mathcal{J}_{0,3}; \ldots; \\ &\qquad (-1)^{(\kappa-1)(b_2-1)} f_s(\beta_{b_2}), \mathcal{J}_{(-1)^{(\kappa-1)b_2} f_s(\beta_{b_2}),3}; \ldots; -s\big) \\ J_{(\kappa,p)_\mathcal{W}} &= \mathrm{diag}(p; \ldots; 0, \mathcal{J}_{0,3}; \ldots; -p) \\ \hat{Q}_{(\kappa,s)_\mathcal{W}} &= \mathrm{diag}\left(1; \ldots; 1, \mathcal{Q}_{(-1)^{(\kappa-1)b_1} f_s'(\beta_{b_1})}; \ldots; 1, 1, \frac{(-1)^{(\kappa-1)\frac{p}{2}+j-1}}{j}, \frac{1}{j^2}; \ldots; \right. \\ &\qquad \left. 1, \mathcal{Q}_{(-1)^{(\kappa-1)b_2} f_s'(\beta_{b_2})}; \ldots; 1\right), \end{aligned} \tag{5.28}$$

for $s = 2j$, $j \in \mathbb{Z}_{2,\frac{p}{2}}$ and

$$\begin{aligned} J_{(\kappa,s)_\mathcal{W}} &= \mathrm{diag}\big(s; \ldots; (-1)^{(\kappa-1)(b_1-1)} f_s(\beta_{b_1}), \mathcal{J}_{(-1)^{(\kappa-1)b_1} f_s(\beta_{b_1}),3}; \ldots; (-1)^{(\kappa-1)(\frac{p}{2}-1)+j}, \\ &\qquad \mathcal{J}^{(1,2)}_{(-1)^{(\kappa-1)\frac{p}{2}+j},3}; \ldots; (-1)^{(\kappa-1)(b_2-1)} f_s(\beta_{b_2}), \mathcal{J}_{(-1)^{(\kappa-1)b_2} f_s(\beta_{b_2}),3}; \ldots; s\big) \\ \hat{Q}_{(\kappa,s)_\mathcal{W}} &= \mathrm{diag}\left(1; \ldots; 1, \mathcal{Q}_{(-1)^{(\kappa-1)b_1} f_s'(\beta_{b_1})}; \ldots; 1, C_2, \frac{2(-1)^{(\kappa-1)\frac{p}{2}+j-1}}{j(j+1)}; \ldots; \right. \\ &\qquad \left. 1, \mathcal{Q}_{(-1)^{(\kappa-1)b_2} f_s'(\beta_{b_2})}; \ldots; 1\right) \end{aligned} \tag{5.29}$$

for $s = 2j+1$, $j \in \mathbb{Z}_{1,\frac{p}{2}-1}$. For $\frac{p}{2(b',p)} \notin \mathbb{N}$, b' is necessarily even and we have

$$J_{(\mathcal{R}_\kappa^{b'})_\mathcal{W}} = \mathrm{diag}(2p; \ldots; 0, \mathcal{J}_{0,3}; \ldots; -2p)$$

$$\hat{Q}_{(\mathcal{R}_\kappa^{b'})_\mathcal{W}} = \mathrm{diag}\bigg(1;\ldots;1,\,\mathcal{Q}_{(-1)^{(\kappa-1)b_1}f'_{p+b'}(\beta_{b_1})};\ldots;1,1,\,\frac{(-1)^{\frac{\kappa p+b'}{2}-1}}{p},\,\frac{1}{p^2};\ldots;\\ 1,\,\mathcal{Q}_{(-1)^{(\kappa-1)b_2}f'_{p+b'}(\beta_{b_2})};\ldots;1\bigg), \tag{5.30}$$

whereas for $\frac{p}{2(b',p)}\in\mathbb{N}$, we have

$$\begin{aligned} J_{(\mathcal{R}_\kappa^{b'})_\mathcal{W}} &= \mathrm{diag}(2p;\ldots;0,\,J_{b',b};\ldots;(-1)^{b'}2p)\\ \hat{Q}_{(\mathcal{R}_\kappa^{b'})_\mathcal{W}} &= \mathrm{diag}(1;\ldots;1,\,Q_{b',b};\ldots;1), \end{aligned} \tag{5.31}$$

where

$$\begin{array}{lll} J_{b',b} = \mathcal{J}_{0,3}^{(1,2)}, & Q_{b',b} = \mathrm{diag}\left(C_2,\,\dfrac{2(-1)^{(\kappa-1)b}}{f''_{p+b'}(\beta_b)}\right), & b = \frac{(2j-1)p}{2(b',p)},\quad j\in\mathbb{Z}_{1,(b',p)}\\ J_{b',b} = \mathcal{J}_{0,3}, & Q_{b',b} = \mathcal{Q}_{(-1)^{(\kappa-1)b}f'_{p+b'}(\beta_b)}, & \text{otherwise.} \end{array} \tag{5.32}$$

Further simplifications are possible but not included here. Above, (n,m) denotes the greatest common divisor of the integers n and m.

6. Generalized Jordan chains and non-canonical Jordan forms

In preparation for the description of the Jordan decompositions of the fusion matrices in terms of modular data in section 7.2, we here present a particularly convenient similarity matrix which brings all the fusion matrices simultaneously to Jordan form. Unlike the similarity transformations considered so far, however, this one converts the fundamental fusion matrix Y into a *non-canonical* Jordan form.

Based on the Jordan chain (5.13), we introduce the vectors

$$\begin{aligned} \tilde{w}_b^{(1)} &= \mu_{b,1}^{(1)}w_b^{(1)}, \qquad \tilde{w}_b^{(2)} = \mu_{b,2}^{(2)}w_b^{(2)} + \mu_{b,1}^{(2)}w_b^{(1)},\\ \tilde{w}_b^{(3)} &= \mu_{b,3}^{(3)}w_b^{(3)} + \mu_{b,2}^{(3)}w_b^{(2)} + \mu_{b,1}^{(3)}w_b^{(1)}, \end{aligned} \tag{6.1}$$

where $\mu_{b,\ell}^{(\ell)}\neq 0$. Unlike the original triplet $w_b^{(1)}, w_b^{(2)}, w_b^{(3)}$, the triplet $\tilde{w}_b^{(1)}, \tilde{w}_b^{(2)}, \tilde{w}_b^{(3)}$ does *not* form a Jordan chain. Instead, it forms the *generalized* Jordan chain

$$Y\big(\tilde{w}_b^{(1)}\ \tilde{w}_b^{(2)}\ \tilde{w}_b^{(3)}\big) = \big(\tilde{w}_b^{(1)}\ \tilde{w}_b^{(2)}\ \tilde{w}_b^{(3)}\big)\begin{pmatrix} \beta_b & \frac{\mu_{b,2}^{(2)}}{\mu_{b,1}^{(1)}} & \frac{\mu_{b,2}^{(2)}\mu_{b,2}^{(3)}-\mu_{b,1}^{(2)}\mu_{b,3}^{(3)}}{\mu_{b,1}^{(1)}\mu_{b,2}^{(2)}}\\ 0 & \beta_b & \frac{\mu_{b,3}^{(3)}}{\mu_{b,2}^{(2)}}\\ 0 & 0 & \beta_b \end{pmatrix}. \tag{6.2}$$

This generalization is particularly useful if the (3×3)-matrix in (6.2) can be written as

$$\begin{aligned} \phi(\mathcal{J}_{\theta_b,3}) = \phi\left(\begin{pmatrix}\theta_b & 1 & 0\\ 0 & \theta_b & 1\\ 0 & 0 & \theta_b\end{pmatrix}\right) &= \begin{pmatrix}\phi(\theta_b) & \phi'(\theta_b) & \frac{1}{2}\phi''(\theta_b)\\ 0 & \phi(\theta_b) & \phi'(\theta_b)\\ 0 & 0 & \phi(\theta_b)\end{pmatrix}\\ &= \begin{pmatrix}\beta_b & -2\sin\theta_b & -\cos\theta_b\\ 0 & \beta_b & -2\sin\theta_b\\ 0 & 0 & \beta_b\end{pmatrix}, \end{aligned} \tag{6.3}$$

where

$$\phi(\theta) = 2\cos\theta. \tag{6.4}$$

We note that setting

$$\mu_{b,\ell}^{(\ell)}=\frac{(-1)^{b-1}(-2\sin\theta_b)^\ell}{p\sqrt{2p}},\qquad \mu_{b,1}^{(2)}=\frac{2(-1)^b\cos\theta_b}{p\sqrt{2p}},\qquad \mu_{b,\kappa}^{(3)}=\frac{(2\kappa-1)(-1)^{b-1}\sin\kappa\theta_b}{p\sqrt{2p}} \tag{6.5}$$

respects (6.3). In this case, and written in the basis (3.1) indicated by the labeling $(\kappa,s)_{\mathcal{W}}$, $\big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}}$, the entries of the vectors $\tilde{w}_b^{(\ell)}$ read

$$\tilde{w}_{b'}^{(1)}=\begin{pmatrix}\frac{2(-1)^{\kappa b'}\sin s\theta_{b'}}{p\sqrt{2p}}\\ \hline 0\end{pmatrix},\qquad \tilde{w}_{b'}^{(2)}=\begin{pmatrix}\frac{2(-1)^{\kappa b'}s\cos s\theta_{b'}}{p\sqrt{2p}}\\ \hline \frac{4(-1)^{(\kappa-1)b'}p\cos b\theta_{b'}}{p\sqrt{2p}}\end{pmatrix},\qquad \tilde{w}_{b'}^{(3)}=\begin{pmatrix}\frac{(-1)^{\kappa b'+1}s^2\sin s\theta_{b'}}{p\sqrt{2p}}\\ \hline \frac{4(-1)^{\kappa b'-b'+1}pb\sin b\theta_{b'}}{p\sqrt{2p}}\end{pmatrix}, \tag{6.6}$$

where there are $2p$ entries above and $2p-2$ entries below the horizontal separator in a given vector. It is likewise convenient to normalize the eigenvectors $v_{(\kappa'-1)p}$ and v_b by introducing the vectors

$$\begin{aligned}\tilde{v}_{(\kappa'-1)p}&=\mu_{(\kappa'-1)p}v_{(\kappa'-1)p}=\frac{(-1)^{(\kappa'-1)(p-1)}}{p\sqrt{2p}}v_{(\kappa'-1)p}=\begin{pmatrix}\frac{s(-1)^{(\kappa'-1)(\kappa p+s)}}{p\sqrt{2p}}\\ \hline \frac{2p(-1)^{(\kappa'-1)((\kappa-1)p+b)}}{p\sqrt{2p}}\end{pmatrix}\\ \tilde{v}_{b'}&=\mu_{b'}v_{b'}=\frac{2(-1)^{b'}\sin\theta_{b'}}{p\sqrt{2p}}v_{b'}=\begin{pmatrix}\frac{2(-1)^{\kappa b'+\kappa-1}\sin s\theta_{b'}}{p\sqrt{2p}}\\ \hline 0\end{pmatrix}.\end{aligned} \tag{6.7}$$

Now, constructed by concatenating the vectors $\tilde{v}_j$ and $\tilde{w}_b^{(\ell)}$, the similarity matrix

$$\tilde{Q}=\Big(\tilde{v}_0\;\Big|\;\tilde{v}_1\quad\tilde{w}_1^{(1)}\quad\tilde{w}_1^{(2)}\quad\tilde{w}_1^{(3)}\;\Big|\;\ldots\;\Big|\;\tilde{v}_{p-1}\quad\tilde{w}_{p-1}^{(1)}\quad\tilde{w}_{p-1}^{(2)}\quad\tilde{w}_{p-1}^{(3)}\;\Big|\;\tilde{v}_p\Big) \tag{6.8}$$

converts X and Y simultaneously

$$\tilde{Q}^{-1}X\tilde{Q}=\tilde{J}_X,\qquad \tilde{Q}^{-1}Y\tilde{Q}=\tilde{J}_Y \tag{6.9}$$

into the Jordan forms

$$\tilde{J}_X=J_X,\qquad \tilde{J}_Y=\mathrm{diag}\big(\beta_0;\,\beta_1,\phi\big(\mathcal{J}_{\theta_1,3}\big);\,\ldots;\,\beta_{p-1},\phi\big(\mathcal{J}_{\theta_{p-1},3}\big);\,\beta_p\big). \tag{6.10}$$

It is noted that $\tilde{J}_Y$ is a non-canonical Jordan form, as already announced. More generally, for $\mathcal{N}=X^{\kappa-1}f(Y)$ as in (5.9), we have

$$\tilde{Q}^{-1}\mathcal{N}\tilde{Q}=\tilde{J}_{\mathcal{N}}=J_X^{\kappa-1}f\circ\phi\big(\mathrm{diag}\big(\theta_0;\ldots;\theta_b,\mathcal{J}_{\theta_b,3};\ldots;\theta_p\big)\big) \tag{6.11}$$

with b running from 1 to $p-1$. That is,

$$\tilde{J}_{\mathcal{N}}=\mathrm{diag}\big(f(\beta_0);\ldots;(-1)^{(\kappa-1)(b-1)}f(\beta_b),(-1)^{(\kappa-1)b}f\circ\phi\big(\mathcal{J}_{\theta_b,3}\big);\ldots;(-1)^{(\kappa-1)p}f(\beta_p)\big), \tag{6.12}$$

where

$$\begin{aligned}f\circ\phi(\mathcal{J}_{\theta_b,3})&=\begin{pmatrix}f\circ\phi(\theta_b)&(f\circ\phi)'(\theta_b)&\frac{1}{2}(f\circ\phi)''(\theta_b)\\0&f\circ\phi(\theta_b)&(f\circ\phi)'(\theta_b)\\0&0&f\circ\phi(\theta_b)\end{pmatrix}\\&=\begin{pmatrix}f(\beta_b)&-2\sin\theta_bf'(\beta_b)&-\cos\theta_bf'(\beta_b)+2\sin^2\theta_bf''(\beta_b)\\0&f(\beta_b)&-2\sin\theta_bf'(\beta_b)\\0&0&f(\beta_b)\end{pmatrix}.\end{aligned} \tag{6.13}$$

It is straightforward, albeit somewhat tedious, to show that the inverse of $\tilde{Q}$ is given by

$$\tilde{Q}^{-1} = \frac{1}{p\sqrt{2p}} \left(\begin{array}{c|c|c} 0 & p & p-b \\ \hline \vdots & \vdots & \vdots \\ \hline (-1)^{\kappa(b'-1)+1}p^2\sin b\theta_{b'} & 0 & \frac{1}{4}(-1)^{(\kappa-1)(b'-1)}p^2\sin b\theta_{b'} \\ (-1)^{\kappa b'}p^2\sin b\theta_{b'} & 0 & \frac{1}{4}(-1)^{(\kappa-1)b'+1}\big(p^2-2(p-b)^2\big)\sin b\theta_{b'} \\ 0 & (-1)^{(\kappa-1)b'}p & (-1)^{(\kappa-1)b'}(p-b)\cos b\theta_{b'} \\ 0 & 0 & (-1)^{(\kappa-1)b'+1}\sin b\theta_{b'} \\ \hline \vdots & \vdots & \vdots \\ \hline 0 & (-1)^{(\kappa-1)p}p & (-1)^{(\kappa-1)p+b}(p-b) \end{array}\right). \tag{6.14}$$

The columns are labeled by $(\kappa, b)_{\mathcal{W}}$, $(\kappa, p)_{\mathcal{W}}$ and $\big(\mathcal{R}_\kappa^b\big)_{\mathcal{W}}$, while the rows are labeled by $\tilde{v}_0$, the $p-1$ quadruplets $\tilde{v}_{b'}$, $\tilde{w}_{b'}^{(1)}$, $\tilde{w}_{b'}^{(2)}$ and $\tilde{w}_{b'}^{(3)}$, and $\tilde{v}_p$. Finally, the determinant of $\tilde{Q}$ follows from that of Q (5.18) and is found to be

$$\det\tilde{Q} = \mu_0\mu_p\left(\prod_{b=1}^{p-1}\mu_b\prod_{\ell=1}^{3}\mu_{b,\ell}^{(\ell)}\right)\det Q = \frac{\det Q}{2^{2p-1}p^{6p-10}} = (-1)^p\left(\frac{8}{p}\right)^{p-1} \tag{6.15}$$

7. Generalized Verlinde formula

7.1. Characters and modular data

The irreducible characters are given by

$$\hat{\chi}_{\kappa,s}(q) = \chi[(\kappa,s)_{\mathcal{W}}](q) = \frac{1}{\eta(q)}\sum_{j\in\mathbb{Z}}(2j+\kappa)q^{p(j+\frac{\kappa p-s}{2p})^2}, \tag{7.1}$$

where $\eta(q)$ is the Dedekind eta function:

$$\eta(q) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m). \tag{7.2}$$

The characters of the indecomposable rank-2 representations are given by

$$\chi\big[\big(\mathcal{R}_1^b\big)_{\mathcal{W}}\big](q) = \chi\big[\big(\mathcal{R}_2^{p-b}\big)_{\mathcal{W}}\big](q) = 2\hat{\chi}_{2,b}(q) + 2\hat{\chi}_{1,p-b}(q). \tag{7.3}$$

There are $p+1$ linearly independent projective characters, namely $\hat{\chi}_{\kappa,p}(q)$ for $\kappa\in\mathbb{Z}_{1,2}$ and the ones in (7.3).

The set of irreducible characters (7.1) does not close under modular transformations. Instead, a representation of the modular group is obtained [31–34] by enlarging the set with the $p-1$ so-called *pseudo-characters*:

$$\hat{\chi}_{0,b}(q) = \mathrm{i}\tau\big(b\hat{\chi}_{1,p-b}(q) - (p-b)\hat{\chi}_{2,b}(q)\big), \tag{7.4}$$

where the modular parameter is given by

$$q = \mathrm{e}^{2\pi\mathrm{i}\tau}. \tag{7.5}$$

Writing the associated (generalized) modular S-matrix in block form with respect to the distinction between proper characters $\hat{\chi}_{\kappa,s}(q)$ and pseudo-characters $\hat{\chi}_{0,b}(q)$, the entries read

$$S = \begin{pmatrix} S_{\kappa,s}^{\kappa',s'} & S_{\kappa,s}^{0,b'} \\ S_{0,b}^{\kappa',s'} & S_{0,b}^{0,b'} \end{pmatrix} = \begin{pmatrix} \frac{(2-\delta_{s',p})(-1)^{\kappa s'+\kappa' s+\kappa\kappa' p} s \cos\frac{ss'\pi}{p}}{p\sqrt{2p}} & \frac{2(-1)^{\kappa b'}\sin\frac{sb'\pi}{p}}{p\sqrt{2p}} \\ \frac{2(-1)^{\kappa' b}(p-s')\sin\frac{bs'\pi}{p}}{\sqrt{2p}} & 0 \end{pmatrix}. \tag{7.6}$$

Here the lower (upper) indices refer to the row (column) labeling. This matrix is not symmetric and not unitary, but satisfies $S^2 = I$. We note that

$$S_{\kappa,s}^{1,p-b} = S_{\kappa,s}^{2,b} \tag{7.7}$$

implying that, under the modular transformation $\tau \to \frac{-1}{\tau}$, the $2p$ irreducible characters transform into linear combinations of the $p+1$ projective characters (with expansion coefficients $S_{\kappa,s}^{\kappa',p}$ and $\frac{1}{2}S_{\kappa,s}^{2,b}$) and the $p-1$ pseudo-characters (with expansion coefficients $S_{\kappa,s}^{0,b}$), only. We also introduce

$$S_{(\mathcal{R}_\kappa^b)_\mathcal{W}}^{\kappa',s'} = 2\big(S_{\kappa,p-b}^{\kappa',s'} + S_{2\cdot\kappa,b}^{\kappa',s'}\big) \tag{7.8}$$

and similarly for related combinations. We finally note that, formally,

$$S_{\kappa,s}^{2,b} = \frac{\partial}{\partial\theta_b} S_{\kappa,s}^{0,b}. \tag{7.9}$$

Alternatively, one can introduce the $2p$-dimensional, τ-dependent (and thus *improper*) S-matrix

$$\mathcal{S} - \mathrm{i}\tau\tilde{\mathcal{S}} \tag{7.10}$$

(here written in calligraphic to distinguish it from the proper S-matrix in (7.6)) obtained by expanding the pseudo-characters in terms of the irreducible characters. Its entries thus read

$$\mathcal{S}_{\kappa,s}^{\kappa',s'} = S_{\kappa,s}^{\kappa',s'}, \qquad \tilde{\mathcal{S}}_{\kappa,s}^{\kappa',s'} = \frac{2(-1)^{\kappa s'+\kappa' s+\kappa\kappa' p}(p-s')\sin\frac{ss'\pi}{p}}{p\sqrt{2p}} \tag{7.11}$$

from which it follows that

$$(p-b)\tilde{\mathcal{S}}_{\kappa,s}^{1,p-b} = -b\tilde{\mathcal{S}}_{\kappa,s}^{2,b} \tag{7.12}$$

and

$$\tilde{\mathcal{S}}_{\kappa,s}^{1,b'} = -(p-b')S_{\kappa,s}^{0,p-b'}, \qquad \tilde{\mathcal{S}}_{\kappa,s}^{2,b'} = (p-b')S_{\kappa,s}^{0,b'}, \qquad \tilde{\mathcal{S}}_{\kappa,s}^{\kappa',p} = 0. \tag{7.13}$$

It is easily seen that an expression can be written in terms of the proper S-matrix S if and only if it can be written in terms of the improper S-matrix $\mathcal{S} - \mathrm{i}\tau\tilde{\mathcal{S}}$. We use exclusively the proper S-matrix S in the following.

7.2. *Generalized Verlinde formula*

The objective here is to express the spectral decompositions of the fusion matrices in terms of the modular data. The various expressions are *not* unique, as indicated by the rather trivial identity $S_{2,1}^{\kappa,p}\big/S_{1,1}^{1,p} = S_{\kappa,1}^{0,1}\big/S_{1,1}^{0,1}$, for example. First, we observe that the eigenvalues can be written as

$$\beta_{(\kappa'-1)p} = \frac{S_{1,2}^{\kappa',p}}{S_{1,1}^{\kappa',p}}, \qquad \beta_{b'} = \frac{S_{1,2}^{0,b'}}{S_{1,1}^{0,b'}} \tag{7.14}$$

and that

$$f_s(\beta_{(\kappa'-1)p}) = \frac{S^{\kappa',p}_{1,s}}{S^{\kappa',p}_{1,1}}, \qquad f_s(\beta_{b'}) = \frac{S^{0,b'}_{1,s}}{S^{0,b'}_{1,1}}, \qquad f_{p+b}(\beta_{(\kappa'-1)p}) = \frac{S^{\kappa',p}_{(\mathcal{R}^b_1)_{\mathcal{W}}}}{S^{\kappa',p}_{1,1}}, \qquad f_{p+b}(\beta_{b'}) = \frac{S^{0,b'}_{(\mathcal{R}^b_1)_{\mathcal{W}}}}{S^{0,b'}_{1,1}}. \tag{7.15}$$

With reference to (6.13), the remaining entries of the Jordan form $\tilde{J}_{\mathcal{N}}$ (6.12) follow from

$$-2\sin\theta_{b'} f'_s(\beta_{b'}) = \frac{S^{0,b'}_{1,1}S^{2,b'}_{1,s} - S^{2,b'}_{1,1}S^{0,b'}_{1,s}}{\left(S^{0,b'}_{1,1}\right)^2}$$

$$-2\sin\theta_{b'} f'_{p+b}(\beta_{b'}) = \frac{S^{0,b'}_{1,1}S^{2,b'}_{(\mathcal{R}^b_1)_{\mathcal{W}}} - S^{2,b'}_{1,1}S^{0,b'}_{(\mathcal{R}^b_1)_{\mathcal{W}}}}{\left(S^{0,b'}_{1,1}\right)^2} = \frac{S^{2,b'}_{(\mathcal{R}^b_1)_{\mathcal{W}}}}{S^{0,b'}_{1,1}}$$

$$-\cos\theta_{b'} f'_s(\beta_{b'}) + 2\sin^2\theta_{b'} f''_s(\beta_{b'}) = \frac{S^{1,p}_{1,s}S^{1,p}_{1,s}S^{0,b'}_{1,s}}{S^{2,p}_{2,1}S^{1,p}_{1,2}S^{0,b'}_{1,1}} + \frac{S^{1,p}_{1,1}S^{0,b'}_{1,s}}{S^{1,p}_{1,2}S^{0,b'}_{1,1}} + \frac{S^{2,p}_{2,1}S^{0,b'}_{1,2}S^{2,b'}_{2,s} + (S^{2,b'}_{1,1})^2S^{0,b'}_{1,s}}{\left(S^{0,b'}_{1,1}\right)^3}$$

$$-\cos\theta_{b'} f'_{p+b}(\beta_{b'}) + 2\sin^2\theta_{b'} f''_{p+b}(\beta_{b'}) = \frac{S^{2,p}_{1,1}S^{1,p}_{1,2}S^{2,p}_{1,b'}S^{0,b'}_{1,b}}{(S^{1,p}_{1,1})^3S^{0,b'}_{1,1}} - \frac{S^{2,b'}_{1,1}S^{2,b'}_{(\mathcal{R}^b_1)_{\mathcal{W}}}}{\left(S^{0,b'}_{1,1}\right)^2}. \tag{7.16}$$

The columns of the similarity matrix $\tilde{Q}$ (6.8) can be written as

$$\tilde{v}_{(\kappa'-1)p} = \begin{pmatrix} S^{\kappa',p}_{(\kappa,s)_{\mathcal{W}}} \\ S^{\kappa',p}_{(\mathcal{R}^b_\kappa)_{\mathcal{W}}} \end{pmatrix}, \qquad \tilde{v}_{b'} = \begin{pmatrix} \frac{S^{\kappa,p}_{2,1}}{S^{1,p}_{1,1}}S^{0,b'}_{(\kappa,s)_{\mathcal{W}}} \\ \frac{S^{\kappa,p}_{2,1}}{S^{1,p}_{1,1}}S^{0,b'}_{(\mathcal{R}^b_\kappa)_{\mathcal{W}}} \end{pmatrix}$$

$$\tilde{w}^{(1)}_{b'} = \begin{pmatrix} S^{0,b'}_{(\kappa,s)_{\mathcal{W}}} \\ S^{0,b'}_{(\mathcal{R}^b_\kappa)_{\mathcal{W}}} \end{pmatrix}, \qquad \tilde{w}^{(2)}_{b'} = \begin{pmatrix} S^{2,b'}_{(\kappa,s)_{\mathcal{W}}} \\ S^{2,b'}_{(\mathcal{R}^b_\kappa)_{\mathcal{W}}} \end{pmatrix}, \qquad \tilde{w}^{(3)}_{b'} = \begin{pmatrix} \frac{S^{1,p}_{1,s}S^{1,p}_{1,s}}{S^{2,p}_{2,1}S^{1,p}_{1,2}}S^{0,b'}_{(\kappa,s)_{\mathcal{W}}} \\ \frac{S^{2,p}_{1,1}S^{1,p}_{1,2}S^{2,p}_{1,b'}}{(S^{1,p}_{1,1})^3}S^{0,b'}_{(\kappa,b)_{\mathcal{W}}} \end{pmatrix}, \tag{7.17}$$

while the entries of $\tilde{Q}^{-1}$ (6.14) follow from

$$\frac{p}{p\sqrt{2p}} = S^{1,p}_{\kappa,p}, \qquad \frac{p-b}{p\sqrt{2p}} = S^{1,p}_{\kappa,p-b}, \qquad \frac{(-1)^{\kappa(b'-1)+1}p^2\sin b\theta_{b'}}{p\sqrt{2p}} = \frac{S^{1,p}_{1,p}S^{1,p}_{1,p}}{S^{1,p}_{1,2}S^{\kappa,p}_{2,1}}S^{0,b'}_{\kappa,b}$$

$$\frac{\frac{1}{4}(-1)^{(\kappa-1)(b'-1)}p^2\sin b\theta_{b'}}{p\sqrt{2p}} = \frac{S^{\kappa,p}_{2,1}\left(S^{1,p}_{1,p}\right)^2}{\left(S^{1,p}_{1,2}\right)^3}S^{0,b'}_{\kappa-1,b}, \qquad \frac{(-1)^{\kappa b'}p^2\sin b\theta_{b'}}{p\sqrt{2p}} = \frac{S^{1,p}_{1,p}S^{1,p}_{1,p}}{S^{1,p}_{1,2}S^{1,p}_{1,1}}S^{0,b'}_{\kappa,b}$$

$$\frac{\frac{1}{4}(-1)^{(\kappa-1)b'+1}\left(p^2-2(p-b)^2\right)\sin b\theta_{b'}}{p\sqrt{2p}} = \frac{S^{1,1}_{1,p}\left(S^{1,p}_{1,p}\right)^2}{\left(S^{1,p}_{1,2}\right)^3}S^{0,b'}_{\kappa,p-b} + \frac{\left(S^{1,p}_{1,p-b}\right)^2}{\left(S^{1,p}_{1,2}\right)^2}S^{0,b'}_{\kappa-1,b} \tag{7.18}$$

$$\frac{(-1)^{(\kappa-1)b'}p}{p\sqrt{2p}} = \frac{S^{1,p}_{1,1}}{S^{1,p}_{1,2}}S^{2,b'}_{\kappa,p}, \qquad \frac{(-1)^{(\kappa-1)b'}(p-b)\cos b\theta_{b'}}{p\sqrt{2p}} = \frac{S^{1,p}_{1,1}}{S^{1,p}_{1,2}}S^{2,b'}_{\kappa,p-b}$$

$$\frac{(-1)^{(\kappa-1)b'+1}\sin b\theta_{b'}}{p\sqrt{2p}} = \frac{S^{1,p}_{1,1}}{S^{1,p}_{1,2}}S^{0,b'}_{\kappa,p-b}, \qquad \frac{(-1)^{(\kappa-1)p}p}{p\sqrt{2p}} = S^{2,p}_{\kappa,p},$$

$$\frac{(-1)^{(\kappa-1)p+b}(p-b)}{p\sqrt{2p}} = S^{2,p}_{\kappa,p-b}.$$

In summary, the announced generalized Verlinde formula reads

$$\mathcal{N} = \tilde{Q}\tilde{J}_{\mathcal{N}}\tilde{Q}^{-1}, \tag{7.19}$$

where $\tilde{Q}$, $\tilde{J}_{\mathcal{N}}$ and $\tilde{Q}^{-1}$ are expressed in terms of the modular data as outlined above.

8. Partition functions

Due to the presence of reducible yet indecomposable representations, the fusion algebra contains more information than needed for the computation of partition functions as the latter are given in terms of characters only. It is therefore natural to try to identify reductions of the fusion algebra which can replace it when considering partition functions. It is the objective here to outline how certain rings of equivalence classes of fusion-algebra generators do the job. As we will see, two simple requirements ensure that the partition functions can be expressed in terms of the ring data.

In a given (possibly logarithmic) CFT, we let $\{\chi_i(q)\}$ denote the set of irreducible characters and $\{F_\mu\}$ the set of generators of the fusion algebra

$$F_\mu \otimes F_\nu = \bigoplus_\lambda N_{\mu,\nu}{}^\lambda F_\lambda, \qquad N_{\mu,\nu}{}^\lambda \in \mathbb{N}_0. \tag{8.1}$$

The character of a fusion generator is obtained by acting on it with χ:

$$\chi[F_\mu](q) := \sum_i f_\mu{}^i \chi_i(q), \qquad f_\mu{}^i \in \mathbb{N}_0. \tag{8.2}$$

This map extends by linearity. We refer to the matrix formed by the coefficients $f_\mu{}^i$ as the *structure* matrix.

We let $\{G_m\}$ denote the set of equivalence classes of the linear span of fusion generators with respect to some equivalence relation $\sim$. The projector onto these classes is denoted by G and maps a fusion generator into a linear combination of equivalence classes:

$$G[F_\mu] := \sum_m h_\mu{}^m G_m, \qquad h_\mu{}^m \in \mathbb{C}. \tag{8.3}$$

This projector extends by linearity with respect to direct sums.

Let a multiplication $*$ be defined on the set of equivalence classes

$$G_m * G_n = \sum_\ell M_{m,n}{}^\ell G_\ell, \qquad M_{m,n}{}^\ell \in \mathbb{C}. \tag{8.4}$$

For this to be compatible with the fusion rules, we *require* that

$$G[F_\mu \otimes F_\nu] = G[F_\mu] * G[F_\nu] \tag{8.5}$$

and subsequently say that the fusion rules (8.1) induce the multiplication rules on the equivalence classes. As a consequence, we have

$$\sum_\lambda N_{\mu,\nu}{}^\lambda h_\lambda{}^\ell = \sum_{m,n} h_\mu{}^m h_\nu{}^n M_{m,n}{}^\ell. \tag{8.6}$$

We also introduce a map $\tilde{\chi}$ from the set of equivalence classes to the set of characters

$$\tilde{\chi}[G_m] := \sum_i g_m{}^i \chi_i(q), \qquad g_m{}^i \in \mathbb{C}, \tag{8.7}$$

requiring that

$$\tilde{\chi} \circ G = \chi. \tag{8.8}$$

This implies that every lift G_m^{-1} of the equivalence class G_m to the set of fusion generators has the same character

$$\chi[G_m^{-1}] = \tilde{\chi}[G_m]. \tag{8.9}$$

From examining $\chi[F_\mu]$, it follows that

$$f_\mu{}^i = \sum_m h_\mu{}^m g_m{}^i. \tag{8.10}$$

We now consider the partition function (to be discussed further in section 8.2)

$$Z_{\mu,\nu}(q) := \chi[F_\mu \otimes F_\nu](q) = \sum_{\lambda,i} N_{\mu,\nu}{}^\lambda f_\lambda{}^i \chi_i(q). \tag{8.11}$$

Using the above, including the two requirements, we see that this can be written in terms of the data for the equivalence classes

$$Z_{\mu,\nu}(q) = \sum_{m,n,\ell,i} h_\mu{}^m h_\nu{}^n M_{m,n}{}^\ell g_\ell{}^i \chi_i(q). \tag{8.12}$$

By construction, we thus have

$$\sum_{m,n,\ell} h_\mu{}^m h_\nu{}^n M_{m,n}{}^\ell g_\ell{}^i \in \mathbb{N}_0. \tag{8.13}$$

It follows that, in order to obtain the partition functions (8.11), it suffices to know the algebra of the equivalence classes provided this algebra respects the two requirements. As already mentioned, this property of the partition functions is the rationale for imposing the two requirements. As we will discuss in the following, when the equivalence classes correspond to the generators of the Grothendieck group of the characters of $\mathcal{WLM}(1,p)$, the induced multiplication rules of the corresponding Grothendieck ring follow straightforwardly from the fusion algebra, and $\tilde{\chi}$ is an almost trivial bijection.

A natural objective is to determine a minimal algebra, that is, one of smallest possible dimension, of equivalence classes compatible with the fusion algebra. A lower bound for this dimension is the number of linearly independent characters appearing in the fusion algebra, where we note that this number can be smaller that the number of irreducible characters. Another interesting problem is to determine the minimal algebra corresponding to a ring over the integers $\mathbb{Z}$. The fusion algebra itself is such a ring with $* = \hat{\otimes}$, so an upper bound on the dimension is given, in this case, by the dimension of the fusion algebra. We hope to address these questions elsewhere.

8.1. Grothendieck ring of $\mathcal{WLM}(1,p)$

The Grothendieck ring of $\mathcal{WLM}(1,p)$ is obtained by elevating the character identities of $\mathcal{WLM}(1,p)$ to identities between the corresponding fusion generators. From (7.3), we thus impose the equivalence relations

$$\big(\mathcal{R}_1^b\big)_{\mathcal{W}} \sim \big(\mathcal{R}_2^{p-b}\big)_{\mathcal{W}} \sim 2(2,b)_{\mathcal{W}} \oplus 2(1,p-b)_{\mathcal{W}}, \qquad b \in \mathbb{Z}_{1,p-1}. \tag{8.14}$$

In terms of equivalence classes, this means that $|\{G_m\}| = 2p$ where

$$G[(\kappa,s)_{\mathcal{W}}] = G_{\kappa,s}, \qquad G\big[\big(\mathcal{R}_1^b\big)_{\mathcal{W}}\big] = G\big[\big(\mathcal{R}_2^{p-b}\big)_{\mathcal{W}}\big] = 2G_{2,b} + 2G_{1,p-b}. \tag{8.15}$$

It is easily verified that (8.5) is respected by the multiplication $*$ whose multiplication rules are given by

$$G_{\kappa,s} * G_{\kappa',s'} = \sum_{j=|s-s'|+1,\text{ by }2}^{p-|p-s-s'|-1} G_{\kappa\cdot\kappa',j} + \sum_{\beta=\epsilon(s+s'-p-1),\text{ by }2}^{s+s'-p-1} (2-\delta_{\beta,0})(G_{\kappa\cdot\kappa',p-\beta} + G_{2\cdot\kappa\cdot\kappa',\beta}), \tag{8.16}$$

where $G_{\kappa,0} \equiv 0$. These rules actually correspond to a transcription of the first fusion rule in (2.4). As already indicated, the map $\tilde{\chi}$ is simply given by the bijection

$$\tilde{\chi}[G_{\kappa,s}] = \chi[(\kappa, s)_{\mathcal{W}}] = \hat{\chi}_{\kappa,s} \tag{8.17}$$

between the set of Grothendieck generators and the set of irreducible characters. The two index sets $\{m\}$ and $\{i\}$ can therefore be identified, and we have

$$g_m{}^i = \delta_m{}^i \quad \Rightarrow \quad f_\mu{}^i = h_\mu{}^i. \tag{8.18}$$

8.2. *Partition functions in $\mathcal{WLM}(1, p)$*

From the lattice description [6] of $\mathcal{WLM}(1, p)$, every indecomposable representation appearing in (2.1) can be associated with a boundary condition. The corresponding characters are in general not linearly independent. We can nevertheless talk about partition functions arising when combining two such boundary conditions as in (8.11). Following the discussion above, we thus have

$$Z_{\mu,\nu}(q) = \sum_{\lambda,i} N_{\mu,\nu}{}^\lambda f_\lambda{}^i \chi_i(q) = \sum_{m,n,i} h_\mu{}^m h_\nu{}^n M_{m,n}{}^i \chi_i(q) \tag{8.19}$$

implying that

$$\sum_\lambda N_{\mu,\nu}{}^\lambda f_\lambda{}^i = \sum_{m,n} h_\mu{}^m h_\nu{}^n M_{m,n}{}^i, \tag{8.20}$$

where the structure matrix $f_\mu{}^m = h_\mu{}^m$ is the $((4p-2) \times 2p)$-dimensional matrix defined by (8.2) or equivalently by (8.3). The explicit form of this matrix follows from (8.17) and (7.3) or equivalently from (8.15). Ordering the rows as in (3.1) and the columns as

$$G_{1,1}, G_{2,1}; \ldots; G_{1,s}, G_{2,s}; \ldots; G_{1,p}, G_{2,p}, \tag{8.21}$$

the structure matrix is given by

$$f_\mu{}^m = h_\mu{}^m = \begin{pmatrix} I_{p-1} & 0 & 0 \\ 0 & I_{p-1} & 0 \\ 0 & 0 & I_2 \\ 2C_{p-1} & 2I_{p-1} & 0 \\ 2I_{p-1} & 2C_{p-1} & 0 \end{pmatrix}. \tag{8.22}$$

8.3. *Relation between Verlinde formulas for $\mathcal{WLM}(1, p)$*

The generalized Verlinde formula derived in section 7.2 yields the multiplicities $N_{\mu,\nu}{}^\lambda$ in (8.19), whereas the multiplicities $M_{m,n}{}^i$, also in (8.19), are given by the generalized Verlinde formulas for the Grothendieck ring appearing in [4, 9]. Here we demonstrate how the so-called Moore–Penrose inverse of $h_\mu{}^m$ allows us to isolate $M_{m,n}{}^i$ from relation (8.20).

First, we recall that for every $n \times m$ matrix A, there is a unique matrix $A^\dagger$ satisfying the four Penrose equations (see [35, 36], for example):

$$AA^\dagger A = A, \qquad A^\dagger A A^\dagger = A^\dagger, \qquad (AA^\dagger)^* = AA^\dagger, \qquad (A^\dagger A)^* = A^\dagger A, \tag{8.23}$$

where A^* denotes the conjugate transpose of A. The matrix $A^\dagger$ is called the Moore–Penrose inverse, or pseudoinverse for short, of A. Clearly, $A^\dagger$ is an $m \times n$ matrix, and if A is nonsingular, then $A^\dagger = A^{-1}$. It also follows readily that $AA^\dagger$ and $A^\dagger A$ are projection matrices. Furthermore, if A has a full column rank, then A^*A is invertible and $A^\dagger = (A^*A)^{-1}A^*$ implying, in particular, that $A^\dagger A = I$.

Now, the structure matrix (8.22) has full *column* rank so

$$\sum_{\mu} h_m^{\dagger\ \mu} h_{\mu}{}^{n} = \delta_m{}^{n}. \tag{8.24}$$

It follows that

$$M_{m,n}{}^{\ell} = \sum_{\mu,\nu,\lambda} h_m^{\dagger\ \mu} h_n^{\dagger\ \nu} N_{\mu,\nu}{}^{\lambda} h_{\lambda}{}^{\ell} = \sum_{\mu,\nu,\lambda} f_m^{\dagger\ \mu} f_n^{\dagger\ \nu} N_{\mu,\nu}{}^{\lambda} f_{\lambda}{}^{\ell}, \tag{8.25}$$

which expresses the output $M_{m,n}{}^{\ell}$ of the generalized Verlinde formula for the Grothendieck ring in terms of fusion data. Isolating $N_{\mu,\nu}{}^{\lambda}$, on the other hand, from (8.20) is not achieved by application of the Moore–Penrose inverse of the structure matrix simply because this structure matrix does not have full *row* rank so

$$\sum_{i} f_{\lambda}{}^{i} f_i^{\dagger\ \mu} \neq \delta_{\lambda}{}^{\mu}. \tag{8.26}$$

9. Conclusion

We have described the graph fusion algebra of $\mathcal{WLM}(1, p)$. The corresponding fusion matrices (adjacency matrices) are mutually commuting, but in general not diagonalizable. Nevertheless, they can be simultaneously brought to Jordan form, albeit typically non-canonical Jordan form, by a similarity transformation. The two fundamental fusion matrices are simultaneously brought to Jordan canonical form by the similarity matrix Q. For every fusion matrix $\mathcal{N}$, we have provided a modified Q-matrix $Q_{\mathcal{N}}$ converting $\mathcal{N}$ to Jordan canonical form. These Jordan canonical forms are given explicitly and consist of Jordan blocks of rank 1, 2 or 3. The various similarity transformations and Jordan forms can be expressed in terms of modular data. This gives rise to a generalized Verlinde formula for the fusion matrices. Its relation to the partition functions in the model is discussed in a general framework. By application of a particular structure matrix and its Moore–Penrose inverse, this Verlinde formula reduces to the Verlinde-like formula [4] for the associated Grothendieck ring.

We recall that fusion graphs have been instrumental in the classification of rational conformal field theories on the cylinder and on the torus. It is our hope that the present work will be a step toward extending these fundamental insights to the logarithmic conformal field theories. First, though, one should extend our results to the general series of $\mathcal{W}$-extended logarithmic minimal models $\mathcal{WLM}(p, p')$. In this direction, we have recently worked out the corresponding graph fusion algebras and determined their spectral decompositions [37]. The next objective is to determine the associated Verlinde-like formulas which we hope to address elsewhere.

From section 8, we recall that the partition functions can be expressed in terms of a ring of equivalence classes of fusion-algebra generators provided two simple requirements are respected. As indicated following (8.13), in particular, there are many interesting problems related to these fusion-algebra compatible rings. The determination of a minimal such ring or the classification of similar rings over the integers are natural examples.

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Appendix A. On the polynomial fusion ring in section 2.2

A.1. Simplification of isomorphism

Here we prove the simplification (2.14) of the isomorphism (2.11). Recalling the notation

$$P_n(x) = (x^2-4)U_{n-1}^3\left(\frac{x}{2}\right), \qquad P_{n,n'}(x,y) = \left(T_n\left(\frac{x}{2}\right) - T_{n'}\left(\frac{y}{2}\right)\right)U_{n-1}\left(\frac{x}{2}\right)U_{n'-1}\left(\frac{y}{2}\right) \tag{A.1}$$

from [10], we consider

$$\begin{aligned} U_{n-1}\left(\frac{x}{2}\right)U_{n'-1}\left(\frac{y}{2}\right) &\equiv \left(1+\left(\left(\frac{x}{2}\right)^2-1\right)U_{n-1}^2\left(\frac{x}{2}\right)\right)U_{n-1}\left(\frac{x}{2}\right)U_{n'-1}\left(\frac{y}{2}\right) \\ &= T_n^2\left(\frac{x}{2}\right)U_{n-1}\left(\frac{x}{2}\right)U_{n'-1}\left(\frac{y}{2}\right) \\ &\equiv U_{n-1}\left(\frac{x}{2}\right)T_{n'}^2\left(\frac{y}{2}\right)U_{n'-1}\left(\frac{y}{2}\right) \\ &= U_{n-1}\left(\frac{x}{2}\right)\left(1+\left(\left(\frac{y}{2}\right)^2-1\right)U_{n'-1}^2\left(\frac{y}{2}\right)\right)U_{n'-1}\left(\frac{y}{2}\right). \end{aligned} \tag{A.2}$$

The first equivalence is modulo $P_n(x)$, the second equivalence is modulo $P_{n,n'}(x,y)$ (applied twice), while the two equalities follow from the identity

$$T_n^2(z) = 1 + (z^2-1)U_{n-1}^2(z). \tag{A.3}$$

As an immediate consequence, we see that

$$0 \equiv U_{n-1}\left(\frac{x}{2}\right)\left(\left(\frac{y}{2}\right)^2-1\right)U_{n'-1}^3\left(\frac{y}{2}\right) \qquad (\mathrm{mod}\ P_n(x),\ P_{n,n'}(x,y)). \tag{A.4}$$

For $n=1$, in which case $U_{n-1}\left(\frac{x}{2}\right)$ is a non-vanishing constant, this implies that

$$P_{n'}(y) \ \equiv\ 0 \qquad (\mathrm{mod}\ P_1(x),\ P_{1,n'}(x,y)). \tag{A.5}$$

A.2. Quotient polynomial ring conditions

Here we complete the verification of the quotient polynomial ring conditions in (2.11). Since

$$(\mathcal{J}_b - \beta_b I),\ (\mathcal{J}_b-\beta_b I)^2 \neq 0, \qquad (\mathcal{J}_b-\beta_b I)^3 = 0, \qquad b\in\mathbb{Z}_{1,p-1}, \tag{A.6}$$

where $\mathcal{J}_b = \mathcal{J}_{\beta_b,3}$ as in (5.16), it follows from the Jordan canonical form J_Y of Y that the minimal polynomial of Y is indeed given by $P_p(Y)$ in (3.9). It also follows that the characteristic polynomial of Y is given as in (3.9). Due to (3.7) and the commutativity of X and Y, the explicit verification of (2.11) is thus completed once we have established that $\tilde{P}_{1,p}(X,Y)=0$ which is equivalent to

$$J_X U_{p-1}\left(\frac{J_Y}{2}\right) = T_p\left(\frac{J_Y}{2}\right)U_{p-1}\left(\frac{J_Y}{2}\right). \tag{A.7}$$

Using (5.17) and (B.5), the left-hand side reads

$$\begin{aligned} J_X U_{p-1}\left(\frac{J_Y}{2}\right) &= \mathrm{diag}\Bigg(U_{p-1}(\alpha_0); U_{p-1}(\alpha_1), -U_{p-1}\left(\frac{\mathcal{J}_1}{2}\right); \ldots; \\ &\qquad (-1)^{b-1}U_{p-1}(\alpha_b), (-1)^b U_{p-1}\left(\frac{\mathcal{J}_b}{2}\right); \ldots; \\ &\qquad (-1)^{p-2}U_{p-1}(\alpha_{p-1}), (-1)^{p-1}U_{p-1}\left(\frac{\mathcal{J}_{p-1}}{2}\right); (-1)^p U_{p-1}(\alpha_p)\Bigg) \\ &= \mathrm{diag}\left(p; 0, (-1)^1 U_{p-1}\left(\frac{\mathcal{J}_1}{2}\right); \ldots; 0, (-1)^{p-1}U_{p-1}\left(\frac{\mathcal{J}_{p-1}}{2}\right); -p\right). \end{aligned} \tag{A.8}$$

Likewise, the right-hand side of (A.7) reads

$$T_p\left(\frac{J_Y}{2}\right)U_{p-1}\left(\frac{J_Y}{2}\right) = \mathrm{diag}\left(p; 0, T_p\left(\frac{\mathcal{J}_1}{2}\right)U_{p-1}\left(\frac{\mathcal{J}_1}{2}\right); \ldots; \right.$$
$$\left. 0, T_p\left(\frac{\mathcal{J}_{p-1}}{2}\right)U_{p-1}\left(\frac{\mathcal{J}_{p-1}}{2}\right); -p\right). \tag{A.9}$$

Using

$$f\left(\frac{\mathcal{J}_b}{2}\right) = \begin{pmatrix} f(\alpha_b) & \frac{1}{2}f'(\alpha_b) & \frac{1}{8}f''(\alpha_b) \\ 0 & f(\alpha_b) & \frac{1}{2}f'(\alpha_b) \\ 0 & 0 & f(\alpha_b) \end{pmatrix} \tag{A.10}$$

for polynomial f, we find

$$T_p\left(\frac{\mathcal{J}_b}{2}\right)U_{p-1}\left(\frac{\mathcal{J}_b}{2}\right) = \begin{pmatrix} (-1)^b & 0 & \frac{1}{8}T_p''(\alpha_b) \\ 0 & (-1)^b & 0 \\ 0 & 0 & (-1)^b \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2}U_{p-1}'(\alpha_b) & \frac{1}{8}U_{p-1}''(\alpha_b) \\ 0 & 0 & \frac{1}{2}U_{p-1}'(\alpha_b) \\ 0 & 0 & 0 \end{pmatrix}$$
$$= (-1)^b U_{p-1}\left(\frac{\mathcal{J}_b}{2}\right) \tag{A.11}$$

and thus recover (A.8). This completes the explicit verification of $\tilde{P}_{1,p}(X,Y) = 0$ and hence of the isomorphism (2.11) already established, using structural and algebraic arguments, in [10].

Appendix B. Properties of the functions $f_k(x)$

Here we derive and list some useful properties of the functions $f_k(x)$ defined in (5.8).

B.1. Recursion relations and special values

For $p > 2$, the functions satisfy recursive relations allowing us to express $xf_k(x)$, for $k \in \mathbb{Z}_{1,2p-2}$, as

$$\begin{aligned} &f_2(x) = xf_1(x) \\ &f_{k-1}(x) + f_{k+1}(x) = xf_k(x), && k \in \mathbb{Z}_{2,p-1} \\ &f_{p+1}(x) = xf_p(x) \\ &2f_p(x) + f_{p+2}(x) = xf_{p+1}(x) \\ &f_{k-1}(x) + f_{k+1}(x) = xf_k(x), && k \in \mathbb{Z}_{p+2,2p-2}. \end{aligned} \tag{B.1}$$

It follows that

$$\begin{aligned} &f_2'(x) = xf_1'(x) + f_1(x), && f_2''(x) = xf_1''(x) + 2f_1'(x) \\ &f_{k-1}'(x) + f_{k+1}'(x) = xf_k'(x) + f_k(x), && f_{k-1}''(x) + f_{k+1}''(x) = xf_k''(x) + 2f_k'(x) \\ &f_{p+1}'(x) = xf_p'(x) + f_p(x), && f_{p+1}''(x) = xf_p''(x) + 2f_p'(x) \\ &2f_p'(x) + f_{p+2}'(x) = xf_{p+1}'(x) + f_{p+1}(x), && 2f_p''(x) + f_{p+2}''(x) = xf_{p+1}''(x) + 2f_{p+1}'(x) \\ &f_{k-1}'(x) + f_{k+1}'(x) = xf_k'(x) + f_k(x), && f_{k-1}''(x) + f_{k+1}''(x) = xf_k''(x) + 2f_k'(x) \end{aligned} \tag{B.2}$$

with the conditions on k adopted from (B.1). It is noted that we have not included any relations involving $xf_{2p-1}(x)$ for general x. Instead, we focus on evaluations at $x = \beta_j$ for $j \in \mathbb{Z}_{0,p}$, where it is recalled that $\beta_j = 2\alpha_j$ with α_j defined in (3.8). We thus find that

$$\begin{aligned}
&2(-1)^i f_p(\beta_j) + f_{2p-2}(\beta_j) = \beta_j f_{2p-1}(\beta_j), && j \in \mathbb{Z}_{1,p-1} \text{ or } i = j \in \{0, p\}\\
&2(-1)^b f_p'(\beta_b) + f_{2p-2}'(\beta_b) = \beta_b f_{2p-1}'(\beta_b) + f_{2p-1}(\beta_b), && b \in \mathbb{Z}_{1,p-1}\\
&2(-1)^b f_p''(\beta_b) + f_{2p-2}''(\beta_b) = \beta_b f_{2p-1}''(\beta_b) + 2f_{2p-1}'(\beta_b), && b \in \mathbb{Z}_{1,p-1.}
\end{aligned} \tag{B.3}$$

In establishing these relations, we use that

$$xf_{2p-1}(x) - f_{2p-2}(x) = 2T_p\left(\frac{x}{2}\right)U_{p-1}\left(\frac{x}{2}\right) \tag{B.4}$$

and that

$$U_{p-1}(\alpha_b) = \frac{1}{p}T_p'(\alpha_b) = 0, \qquad T_p(\alpha_j) = \cos(p\theta_j) = (-1)^j \tag{B.5}$$

recalling from (3.8) that $\theta_j = j\pi/p$.

At special values, the evaluation of the functions f_k and their derivatives can be simplified. Some of these results are collected here for simple reference. Additional expressions are found in appendix B.2. We have

$$f_s(\pm 2) = s(\pm 1)^{s-1}, \qquad f_{p+b'}(\pm 2) = 2p(\pm 1)^{p+b'-1}, \qquad f_p(\beta_b) = f_{p+b'}(\beta_b) = 0 \tag{B.6}$$

and

$$\begin{aligned}
&f_s(0) = 0, && f_s'(0) = j(-1)^{j-1}, && f_s''(0) = 0, && s = 2j\\
&f_s(0) = (-1)^j, && f_s'(0) = 0, && f_s''(0) = j(j+1)(-1)^{j-1}, && s = 2j+1
\end{aligned} \tag{B.7}$$

and for p odd

$$\begin{aligned}
&f_{p+b'}(0) = 0, && f_{p+b'}'(0) = b'(-1)^{\frac{p+1}{2}+i}, && f_{p+b'}''(0) = 0, && b' = 2i-1\\
&f_{p+b'}(0) = 2(-1)^{\frac{p-1}{2}+i}, && f_{p+b'}'(0) = 0, && f_{p+b'}''(0) = \tfrac{p^2+4i^2-1}{2}(-1)^{\frac{p+1}{2}+i}, && b' = 2i
\end{aligned} \tag{B.8}$$

and for p even

$$\begin{aligned}
&f_{p+b'}(0) = 0, && f_{p+b'}'(0) = 0, && f_{p+b'}''(0) = b'p(-1)^{\frac{p}{2}+i}, && b' = 2i-1\\
&f_{p+b'}(0) = 0, && f_{p+b'}'(0) = p(-1)^{\frac{p}{2}+i-1}, && f_{p+b'}''(0) = 0, && b' = 2i.
\end{aligned} \tag{B.9}$$

B.2. Trigonometric expressions

The Chebyshev polynomials and their derivatives appearing in Q are evaluated at trigonometric values:

$$U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}, \qquad U_{n-1}(\pm 1) = n(\pm 1)^{n-1}. \tag{B.10}$$

We thus have

$$\begin{aligned}
&f_s(\beta_{b'}) = \frac{\sin s\theta_{b'}}{\sin\theta_{b'}}, \qquad f_s'(\beta_{b'}) = \frac{\sin s\theta_{b'}\cos\theta_{b'} - s\cos s\theta_{b'}\sin\theta_{b'}}{2\sin^3\theta_{b'}}\\
&f_s''(\beta_{b'}) = \frac{\sin s\theta_{b'}(1+2\cos^2\theta_{b'}) - \frac{3}{2}s\cos s\theta_{b'}\sin 2\theta_{b'} - s^2\sin s\theta_{b'}\sin^2\theta_{b'}}{4\sin^5\theta_{b'}}
\end{aligned} \tag{B.11}$$

and

$$f_{p+b}(\beta_{b'}) = 0, \qquad f'_{p+b}(\beta_{b'}) = \frac{(-1)^{b'-1} p \cos b\theta_{b'}}{\sin^2 \theta_{b'}}$$
$$f''_{p+b}(\beta_{b'}) = \frac{(-1)^{b'-1} p\big(b \sin b\theta_{b'} \sin\theta_{b'} + \frac{3}{2}\cos b\theta_{b'} \cos\theta_{b'}\big)}{\sin^4 \theta_{b'}}. \tag{B.12}$$

B.3. Classification of zeros

For $k \in \mathbb{Z}_{1,2}$, we have

$$f'_1(x) = f''_1(x) = 0, \qquad f'_2(x) = 1, \quad f''_2(x) = 0 \tag{B.13}$$

while for $k = s \in \mathbb{Z}_{3,p}$, we have

$$f'_s(\beta_b) = 0 \Leftrightarrow \alpha_b = 0, \qquad s \text{ odd} \Leftrightarrow b = \frac{p}{2}, \qquad s = 2j+1, \quad j \in \mathbb{Z}_{1,\frac{p}{2}-1} \tag{B.14}$$

and

$$f''_s(\beta_b) = 0 \Leftrightarrow \alpha_b = 0, \qquad s \text{ even} \Leftrightarrow b = \frac{p}{2}, \qquad s = 2j, \qquad j \in \mathbb{Z}_{2,\frac{p}{2}}. \tag{B.15}$$

For $k = p + b' \in \mathbb{Z}_{p+1,2p-1}$, we have

$$f'_{p+b'}(\beta_b) = T'_{b'}(\alpha_b)U_{p-1}(\alpha_b) + T_{b'}(\alpha_b)U'_{p-1}(\alpha_b) = \cos\frac{bb'\pi}{p}U'_{p-1}(\alpha_b). \tag{B.16}$$

From (B.14), it then follows that

$$f'_{p+b'}(\beta_b) = 0 \Leftrightarrow \cos\frac{bb'\pi}{p} = 0 \Leftrightarrow 2bb' = mp, \qquad m \text{ odd}. \tag{B.17}$$

Since m is odd, the last identity implies that $2(b,p)$, where (b,p) denotes the greatest common divisor of b and p, is a divisor of p and hence that $\frac{b}{(b,p)}$ is odd. From

$$b'\frac{b}{(b,p)}2(b,p) = m\frac{p}{2(b,p)}2(b,p), \qquad \left(\frac{b}{(b,p)}, \frac{p}{2(b,p)}\right) = 1, \tag{B.18}$$

it then follows that b' is an odd multiple of $\frac{p}{2(b,p)}$. Since $b' \in \mathbb{Z}_{1,p-1}$, we thus have

$$b' = \frac{(2j-1)p}{2(b,p)}, \qquad j \in \mathbb{Z}_{1,(b,p)}, \qquad \frac{p}{2(b,p)} \in \mathbb{Z}_{1,\frac{p}{2}}. \tag{B.19}$$

That is, for given b, $f'_{p+b'}(\beta_b) = 0$ if and only if b' is of the form (B.19) and $\frac{p}{2(b,p)} \in \mathbb{Z}_{1,\frac{p}{2}}$. Due to the symmetric conditions $b, b' \in \mathbb{Z}_{1,p-1}$ and $\cos\frac{bb'\pi}{p} = 0$, we likewise have that, for given b',

$$f'_{p+b'}(\beta_b) = 0 \Leftrightarrow b = \frac{(2j-1)p}{2(b',p)}, \qquad j \in \mathbb{Z}_{1,(b',p)}, \qquad \frac{p}{2(b',p)} \in \mathbb{Z}_{1,\frac{p}{2}}. \tag{B.20}$$

Finally, still for $k = p + b' \in \mathbb{Z}_{p+1,2p-1}$, we have

$$f''_{p+b'}(\beta_b) = T'_{b'}(\alpha_b)U'_{p-1}(\alpha_b) + \tfrac{1}{2}T_{b'}(\alpha_b)U''_{p-1}(\alpha_b) \tag{B.21}$$

and

$$f''_{p+b'}(\beta_b) = 0 \Leftrightarrow \alpha_b = 0, \quad p + b' \text{ even} \Leftrightarrow b = \frac{p}{2}, \quad b' = 2j, \quad j \in \mathbb{Z}_{1,\frac{p}{2}-1}. \tag{B.22}$$

We observe that $f'_k(\beta_b) = f''_k(\beta) = 0$ if and only if $k = 1$.

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